30 Aug 2023

LP Relaxation + Primal–Dual
Min–Cost Matching

Announcement: See Ed Dissussions for link to homework partner form. Fill it out by Fri, 9/1.

If G is a bipartite graph with edge costs $c(u,v)$

M a matching in G

y a vector assigning a real value to each vertex

we say y is compatible with M (M-compatible) if

1. $y_u + y_v \le c(u,v) \quad \forall (u,v) \in E$
2. $y_u + y_v = c(u,v) \quad \forall (u,v) \in M$
3. $y_u = \max_{w \in L} \{y_w\}$ whenever $u \in L \cap F$
4. $y_v = \max_{w \in R} \{y_w\}$ whenever $v \in R \cap F$

Lemma. If M is a matching with k edges and it has a compatible labeling, y, then M has minimum cost among all k-edge matchings.

Proof. Let $Y = \sum_{u \in V} y_u$.

For any matching M', with free vertex set F'

$$\mathrm{cost}(M') = \sum_{(u,v) \in M'} c(u,v)$$

$$\ge \sum_{(u,v) \in M'} (y_u + y_v) = Y - \sum_{u \in F'} y_u$$

For $M' = M$, this is equality!

RHS is a lower bound on cost (M').

As M' ranges over k-edge matchings, the lower bound is smallest when its free vertices have largest labels, i.e. when $M' = M$.

Furthermore when $M' = M$ its cost matches the lower bound.

$$c^y(u,v) = \begin{cases} c(u,v) - y_u - y_v & \text{if } (u,v) \notin M \\ y_u + y_v - c(u,v) & \text{if } (u,v) \in M \end{cases}$$

Primal–Dual Algorithm for Min-Cost Matching

Initialize $M = \emptyset$, $y_u = \min_e \{c(e)\}$ for $u \in R$,

$y_u = 0$ for $u \in L$

while M has free vertices

calculate $c^y(u,v)$ for each edge (u,v).

calculate residual graph G_M

let P be a min-reduced-cost path in G_M from $L \cap F$ to $R \cap F$

$M \leftarrow M \oplus P$ $\quad d_u :=$ min reduced cost of a path in G_M from $L \cap F$ to u.

$y_u \leftarrow y_u + (c^y(P) - d_u)^+$ for $u \in L$

$y_v \leftarrow y_v - (c^y(P) - d_v)^+$ for $v \in R$

$z^+ = \max\{0, z\}$.

endwhile

output M.

Running time: $\frac{n}{2}$ iterations $\times$ $O(m + n \log n)$ per iteration

$= O(mn + n^2 \log n)$.

Correctness: According to lemma above, it boils down to showing if M, y are compatible at start of loop iter, the new M, y are compatible at the end.

Let M', y' be new matching, labeling.

M, y the matching & labeling at start.

Properties ① & ②:

Case 1. $\overset{(u,v)}{e} \in M$ $\qquad c(u,v) = y_u + y_v$

$$y'_u = y_u + (c^y(P) - d_u)^+$$

$$y'_v = y_v - (c^y(P) - d_v)^+$$

In G_M there is only one edge pointing into u namely (v,u).

$c^y(v,u) = 0$

Every path to u goes thru edge (v,u) with zero reduced cost $\Longrightarrow d_u = d_v$.

$$\Longrightarrow \quad c(u,v) = y'_u + y'_v.$$

Case 2. $e \in P \setminus M$. $\qquad c(u,v) \geq y_u + y_v$.

$$y'_u = y_u + (c^y(P) - d_u)^+ = y_u + c^y(P) - d_u$$

$$y'_v = y_v - (c^y(P) - d_v)^+ = y_v - c^y(P) + d_v$$

$$y'_u + y'_v = y_u + y_v + d_v - d_u$$

$$= y_u + y_v + c^y(u,v)$$

$$= c(u,v)$$