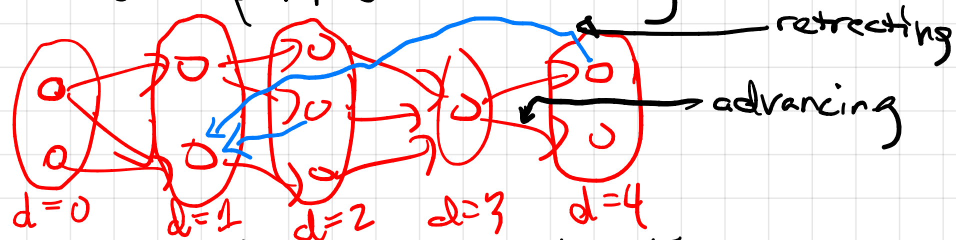
25 Aug 2023

Finishing Hopcroft–Karp

Starting min-cost perfect matching



RECAP of last lecture...

If we do BFS of G_M starting from $L \cap F$ we can label each vertex v with the length of shortest path in G_M from $L \cap F$ to v. Call this $d(v)$.

Edges of G_M, say (u,v), are either

"advancing": $d(v) = d(u) + 1$

or

"retreating": $d(v) < d(u)$

Def. A blocking set of augmenting paths is a (setwise) maximal collection of vertex-disjoint advancing augmenting paths. (Those composed of advancing edges.)

H–K Algorithm

initialize $M = \emptyset$.

while G has an M-augmenting path:

  let $P_1, \dots, P_k$ be a blocking set of augmenting paths.

  $M \leftarrow M \oplus (P_1 \cup \dots \cup P_k)$

endwhile

output M

Lemma. If the shortest M-augmenting path has length ℓ at the start of an iteration of the H–K algorithm, then at the end of that iteration the shortest augmenting path length is $> \ell$.

Proof. Let M' be the matching at the end of that iteration and let P be a shortest M'-augmenting path.

$P = u_0, u_1, u_2, \dots, u_r$

We know u_0, u_r are free in M'.

$\therefore$ they are also free in M.

(a vertex, once matched in the algorithm never becomes free)

Consider the sequence $d(u_0), d(u_1), \dots, d(u_r)$.

For each pair (u_i, u_{i+1}) either the edge $(u_i, u_{i+1}) \in E(G_M)$ or $(u_{i+1}, u_i) \in E(G_M)$.

The case $(u_{i+1}, u_i) \in E(G_M)$ only happens if (u_i, u_{i+1}) was in the blocking set of augmenting paths.

Then $d(u_{i+1}) = d(u_i) + 1$. $\therefore$ (u_i, u_{i+1}) advancing

The case $(u_i, u_{i+1}) \in E(G_M)$

Then $d(u_{i+1}) \le d(u_i) + 1$. $\leftarrow$ equality if and only if (u_i, u_{i+1}) is advancing.

$d(u_r) \le r$ by inductively applying this inequality.

$d(u_r) \ge \ell$ because u_r is free in M, and $d(v) \ge \ell$ for all $v \in R \cap F$ by assumption.

$\therefore$ $r \ge \ell$.

The only way $r = \ell$ could hold is if $d(u_{i+1}) = d(u_i) + 1$ for all $i = 0, 1, \dots, r-1$.

Then all edges (u_i, u_{i+1}) belong the $E(G_M)$ and are advancing.

$\Rightarrow$ P is an advancing M-augmenting path.

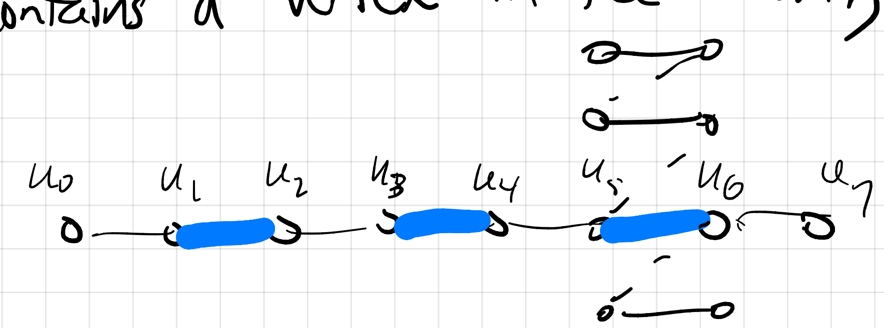
$\Rightarrow$ [maximality of blocking set] $V(P)$ intersects $V(\text{blocking set})$.

endpoints of P are free in M'

all of these are matched in M'

The interior of P contains a vertex in the blocking set, say u.

Path P:



The edge of M' containing u (e.g. edge u_5, u_6 in our diagram) can have only one orientation in G_M. But the proof requires both orientations. $\lightning$

Running time analysis

How many loop iterations occur before the shortest M-augmenting path length exceeds $\sqrt{n}$?

Ans. At most $\frac{1}{2}\sqrt{n}$.

(Shortest path length starts at 1, increases by at least 2 in each iteration.)

How many loop iterations occur after the shortest M-augmenting path length exceeds $\sqrt{n}$?

If M^* is a maximum matching, $k := |M^*| - |M|$ then $M^* \oplus M$ has at least k vertex-disjoint M-aug paths, and they all have $> \sqrt{n}$ vertices.

So $k \cdot \sqrt{n} < n \Rightarrow k < \sqrt{n}$.

Each iteration after that increases size of M by at least 1 $\Longrightarrow$ at most $\sqrt{n}$ iterations remain.

Total. H-K alg has $\leq \frac{3}{2}\sqrt{n}$ outer loop iterations.

Running time $O(m\sqrt{n})$.