Analysis of Algorithms	**Matroid Duality**
CS6820 Fall 2022	**Friday, September 2, 2022**

Theorem The dual matroid is a matroid.

Proof. Let $M = (S, \mathcal{I})$ be matroid. A *basis* of M is a maximal independent set in $\mathcal{I}$. The dual of M is $M^* = (S, \mathcal{I}^*)$, where

$$\mathcal{I}^* = \{A \subseteq S \mid I \cap A = \varnothing \text{ for some basis } I \text{ of } M\}.$$

Note that $M^{**} = M$ and that the bases of M^* are the complements of the bases of M. We show that M^* is a matroid. It is clearly closed downward under $\subseteq$, so we need only show

$$\forall A, B \in \mathcal{I}^* \ \ |A| < |B| \ \Rightarrow\ \exists x \in B - A \ \ A \cup \{x\} \in \mathcal{I}^*.$$

Suppose $A, B \in \mathcal{I}^*$ and $|A| < |B|$. There exist bases I, J of M such that $A \cap J = \varnothing$ and $B \cap I = \varnothing$. Complete $I - A$ to a basis K of M by adding elements of J. Then $A \cap K = \varnothing$ and

$$|K| = |K - I| + |K \cap I| = |K - I| + |I - A| = |K - I| + |I| - |I \cap A|,$$

and $|K| = |I|$, so $|K - I| = |I \cap A|$ and

$$|K \cap (B - A)| \le |K - I| = |I \cap A| \le |A - B| < |B - A|.$$

There must exist $x \in (B - A) - K$, so $(A \cup \{x\}) \cap K = \varnothing$, which says that $A \cup \{x\} \in \mathcal{I}^*$. □

Theorem Cuts in M are cycles in M^* and vice versa.

Proof. By duality, we only need to show one of the two statements.

$$\begin{aligned}
A \text{ is a cut of } M &\Leftrightarrow A \text{ is a minimal set intersecting all bases } I \text{ of } M \\
&\Leftrightarrow A \text{ is a minimal set intersecting all } S - J \text{ for bases } J \text{ of } M^* \\
&\Leftrightarrow A \text{ is a minimal set not contained in any basis } J \text{ of } M^* \\
&\Leftrightarrow A \text{ is a minimal dependent set in } M^* \\
&\Leftrightarrow A \text{ is a cycle of } M^*.
\end{aligned}$$

□

Corollary The blue (respectively, red) rule of M is the red (respectively, blue) rule of M^* with the order of the weights reversed.