Analysis of Algorithms	**LP Dual Fitting**
CS6820 Fall 2022	**Wednesday, November 30, 2022**

1 Weighted Set Cover via LP Dual Fitting

Weighted vertex cover is a special case of the *weighted set cover* problem. We have previously seen an approximation algorithm for weighted set cover, where the approximation ratio involved the function

$$H(m) \ = \ \sum_{j=1}^{m} \frac{1}{j} \ \leq \ 1 + \ln(m). \tag{1}$$

We have run into the function H before, in the analysis of random treaps. In this lecture we show how this result can be obtained by an analysis technique known as *LP dual fitting*.

Recall that in the weighted set cover problem, we are given a set U of n elements along with a set $\mathcal{S}$ of subsets of U with nonegative weights $w : \mathcal{S} \to \mathbb{R}_+$ such that $\bigcup \mathcal{S} = U$; that is, the union of all the sets in $\mathcal{S}$ covers U. The goal is to choose a subcollection $\mathcal{I} \subseteq \mathcal{S}$ of minimum total weight $\sum_{S \in \mathcal{I}} w_S$ such that $\bigcup \mathcal{I} = U$. The decision version of this problem is NP-complete.

Recall that our greedy approximation algorithm chooses sets according to a "minimum weight per new element covered" heuristic. The algorithm constructs $\mathcal{I}$ inductively according to this heuristic. The variable T keeps track of the set of elements not yet covered by $\bigcup \mathcal{I}$.

Algorithm 1 Greedy algorithm for set cover

1: Initialize $\mathcal{I} \leftarrow \varnothing$ and $T \leftarrow U$
2: **while** $T \neq \varnothing$ **do**
3: $\quad S \leftarrow \arg\min_S \{ w(S)/|T \cap S| \mid S \in \mathcal{S},\ T \cap S \neq \varnothing \}$
4: $\quad \mathcal{I} \leftarrow \mathcal{I} \cup \{S\}$
5: $\quad T \leftarrow T \setminus S$
6: **end while**
7: **return** $\mathcal{I}$

Line 3 greedily selects the set minimizing the added weight per new element covered. This quantity might be called the *cost-effectiveness* of the set.

It is clear that the algorithm can be implemented in polynomial time and produces a valid set cover. We wish to show that it achieves an approximation ratio of α, where

$$\alpha = H(\max_{S \in \mathcal{S}} |S|),$$

where the function H is defined in (1).

To analyze the approximation ratio, we will use the LP relaxation of set cover and its dual.

$$\begin{array}{lll} \text{minimize} & \sum_{S\in\mathcal{S}} w_S x_S & \\ \text{subject to} & \sum_{j\in S} x_S \geq 1 & \text{for } j \in U \\ & x_S \geq 0 & \text{for } S \in \mathcal{S} \end{array} \tag{2}$$

$$\begin{array}{lll} \text{maximize} & \sum_{j\in U} y_j & \\ \text{subject to} & \sum_{j\in S} y_j \leq w_S & \text{for } S \in \mathcal{S} \\ & y_j \geq 0 & \text{for } j \in U. \end{array} \tag{3}$$

It will be helpful to add some extra lines to the program that do not influence the choice of sets to put into $\mathcal{I}$, but merely record some extra data relevant to the analysis. Specifically, we compute a vector z indexed by elements of U. The vector z is not a feasible solution of the dual LP, but at the end we will scale it down by a factor of α to obtain $y = z/\alpha$ that is feasible for the dual LP. The scale factor α will be an upper bound on the approximation ratio. This method is sometimes called *dual fitting*.

Algorithm 2 Greedy algorithm for set cover

1: Initialize $\mathcal{I} \leftarrow \varnothing$ and $T \leftarrow U$
2: **while** $T \neq \varnothing$ **do**
3:     $S \leftarrow \arg\min_S \{w_S/|T \cap S| \mid S \in \mathcal{S},\ T \cap S \neq \varnothing\}$
4:     $\mathcal{I} \leftarrow \mathcal{I} \cup \{S\}$
5:     **for** $j \in T \cap S$ **do**
6:         $z_j \leftarrow w_S/|T \cap S|$
7:     **end for**
8:     $T \leftarrow T \setminus S$
9: **end while**
10: $y \leftarrow z/\alpha$
11: **return** $\mathcal{I}$

The following loop invariant is easily shown to hold initially and to be preserved by the body of the while loop:

$$\sum_{j\in U} z_j = \sum_{S\in\mathcal{I}} w_S.$$

Also, note that each z_j is assigned exactly once in line 6, at the time when j becomes covered. The weight of the set S chosen in line 3 is apportioned equally among all the new points covered, and the value assigned to z_j is the portion borne by j.

We will show below (Lemma 1) that the vector y created in line 10 is feasible for the dual LP (3). From this it follows that the approximation ratio is bounded above by α:

$$\sum_{S \in \mathcal{J}} w_S \;=\; \sum_{j \in U} z_j \;=\; \alpha \sum_{j \in U} y_j \;\leq\; \alpha \sum_{S \in \mathrm{OPT}} w_S,$$

where the last line follows from weak duality.

Lemma 1. *The vector y computed in line 10 of Algorithm 2 is feasible for the dual linear program* (3).

Proof. Clearly $y_j \geq 0$ for all j, so we only need to show that $\sum_{j \in S} y_j \leq w_S$ for every set S; equivalently,

$$\sum_{j \in S} z_j \leq \alpha w_S$$

for every set S. Let $m = |S|$ and denote the elements of S by $s_0, s_1, \ldots, s_{m-1}$, where the numbering corresponds to the order in which nonzero values were assigned to the variables z_{s_j} in line 6. This is also the order in which the elements were first covered by a set chosen in line 3.

At the time s_0 was covered and the value z_{s_0} assigned, all of the elements of S still belonged to T. At that time, the cost-effectiveness of S (weight of S divided by number of new elements that would be covered by choosing S) was judged to be w_S/m. The algorithm chose a set with the same or better cost-effectiveness, and z_{s_0} was set equal to the cost-effectiveness of the chosen set; thus

$$z_{s_0} \leq \frac{w_S}{m}. \tag{4}$$

In general, for any $k < m$, at the time s_k was covered and the value z_{s_k} assigned, all of the elements $s_k, s_{k+1}, \ldots, s_{m-1}$ still belonged to T. At that time, the cost-effectiveness of S was judged to be at most $w_S/(m-k)$. The algorithm chose a set with the same or better cost-effectiveness, and z_{s_k} was set to the cost-effectiveness of the chosen set; thus

$$z_{s_k} \leq \frac{w_S}{m-k}. \tag{5}$$

Summing the bounds (5) for $k = 0, \ldots, m-1$, we see that

$$\sum_{j \in S} z_j \;\leq\; \left(\frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{2} + 1 \right) w_S \;=\; H(m) w_S \;\leq\; \alpha w_S,$$

as desired. □