Algorithms	**Problem Set 5**
CS 6820 Fall 2009	**Due November 11, 2009**

Recall the general instructions for handing in homework:

- If possible, please typeset the homework (i.e. format your solutions as an electronic file using latex or Word with mathematical notation).
- Homework solutions done electronically can be handed in by directly uploading them to CMS. Please mail Ashwin (`ashwin85@cs.cornell.edu`) if you have any trouble with this.

(1) *(KT Exercise 10.4)* We say that a graph $G = (V, E)$ is a *triangulated cycle graph* if it consists of the vertices and edges of a triangulated convex n-gon in the plane — in other words, if it can be drawn in the plane as follows.

The vertices are all placed on the boundary of a convex set in the plane (we may assume on the boundary of a circle), with each pair of consecutive vertices on the circle joined by an edge. The remaining edges are then drawn as straight line segments through the interior of the circle, with no pair of edges crossing in the interior. If we let S denote the set of all points in the plane that lie on vertices or edges of the drawing, then each bounded component of the plane after deleting S is bordered by exactly three edges. (This is the sense in which the graph is a "triangulation.")

A triangulated cycle graph is pictured below.

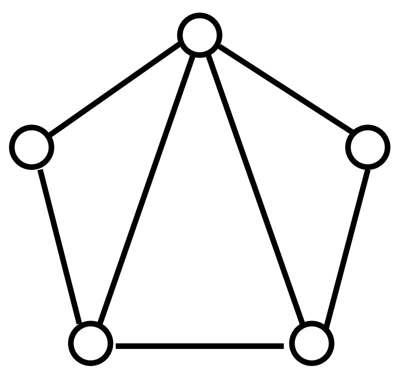

Figure 1: A sample triangulated cycle graph.

Prove that every triangulated cycle graph has a tree decomposition of width at most 2, and describe an efficient algorithm to construct such a decomposition.

(2) *(KT Exercise 10.7)* The *chromatic number* of a graph G is the minimum k such that it has a k-coloring. As we saw in class, it is NP-complete for $k \geq 3$ to decide whether a given input graph has chromatic number $\leq k$.

(a) Show that for every natural number $w \geq 1$, there is a number $k(w)$ so that the following holds. If G is a graph of tree-width at most w, then G has chromatic number at most $k(w)$. (The point is that $k(w)$ depends only on w, not on the number of nodes in G.)

(b) Given an undirected n-node graph $G = (V, E)$ of tree-width at most w, show how to compute the chromatic number of G in time $O(f(w) \cdot p(n))$, where $p(\cdot)$ is a polynomial but $f(\cdot)$ can be an arbitrary function.

(3) Suppose there's a grocery store that you visit on a weekly basis, so that you know the layout well. When you show up with a shopping list, consisting of things to buy, it's natural to ask in what order you should pick up the items on the list so as to minimize the distance you have to travel through the store.

At first glance this seems like it should be a hard problem, since it corresponds to the NP-hard *Traveling Salesman Problem (TSP).* But grocery stores are laid out in such a way as to produce a particularly tractable instance of the TSP — a typical grocery store is organized into aisles, with connections between the aisles occuring only at the front and back of the store. (For purposes of this problem we'll ignore more complex layouts such as the one you find at Wegman's, with passages connecting the middles of aisles; but you're welcome to think about the problem in this case as well if you want.)

With this in mind, let's formalize the *Grocery-Store TSP* as follows. We say that an $n \times n$ *aisle graph* consists of a set of n-node paths $P_1, P_2, \ldots, P_n$ (the "aisles") such that the first node of P_i is connected to the first nodes of P_{i-1} and P_{i+1} (when these are defined), and likewise the last node of P_i is connected to the last nodes of P_{i-1} and P_{i+1} (when these are defined). Figure 2 depicts an 8×8 aisle graph.

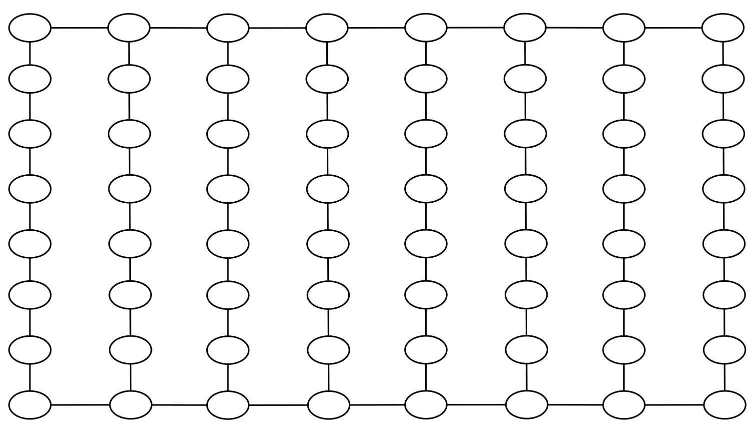

Figure 2: An 8×8 aisle graph.

Now, in an instance of the Grocery-Store TSP, we are given an $n \times n$ aisle graph G, together with a subset S of the nodes in G, and we want to find the shortest cycle in G that contains all the nodes in S together with the first node in the path P_n (the node in the lower-right corner of G, corresponding to the entrance of the store).

Give a polynomial-time algorithm for the Grocery-Store TSP.