CS 6815 Pseudorandomness and Combinatorial Constructions **Fall 2019**

Lecture 9: September 26

Lecturer: Eshan Chattopadhyay *Scribe: Yanyi Liu*

9.1 Introduction

In this lecture, we talked about zig-zag product of graphs, and introduced how to make use of products we covered so far to construct mildly and strongly explicit expanders. Zig-zag product could also be used in the proof for PCP theorem.

9.2 Zig-zag Product

Definition 9.1 *Let G be a (N_1, D_1)-graph, i.e., a D_1-regular graph with N_1 nodes, and H be a (D_1, D_2)-graph. The zig-zag product of G and H is a $(N_1 \times D_1, D_2^2)$-regular graph, which is denoted by $G Ⓩ H$. Let (v, i) where $v \in V(G), i \in [D_1]$ denote a node in $G Ⓩ H$, and the (a, b)-th neighbor of (v, i) in $G Ⓩ H$ where $a, b \in [D_2]$ denote the $(aD_2 + b)$-th neighbor of (v, i) in $G Ⓩ H$. For each node $(v, i) \in G Ⓩ H$, it has D_2^2 neighbors, and the (a, b)-th neighbor is (u, j) computed as follow:*

- *Let i' be the a-th neighbor of node i in H.*
- *Let u be the i'-th neighbor of v in G. Thus, (v, u) is the i'-th edge leaving v.*
- *For u, v is also a neighbor of u. Suppose v is the j'-th neighbor of u.*
- *Let j be the b-th neighbor of j' in H.*

Informally, zig-zag product decreases the degree without hurting expansion too much. In particular, we choose $D_2^2 \ll D_1$. We can consider G as the "large graph" and H as the "small graph". The intuition behind this construction is that we want that a random step on $G Ⓩ H$ corresponds to a step on G, but using a random step on H to choose which neighbor to go. For each $v \in V(G)$, we replace it by a cloud of D_1 nodes, with ith node corresponding to the ith edge that is incident on v. Consider a random step on $G Ⓩ H$, starting from (v, i). It will first jump into a random node in this cloud, i.e., (v, i'), since a is randomly chosen. After i' is determined, we will jump to a deterministic node (u, j') in the cloud of u. Then we will pick up b randomly, which pushes us to a random node (u, j) in that cloud.

An example of zig-zag product is shown in Fig 9.1. The figure was borrowed from lecture note of last semester.

Lemma 9.2 *If H is a complete graph, $G Ⓩ H = G \otimes H$.*

Theorem 9.3 *$G Ⓩ H$ is a D_2^2-regular graph on $N_1 D_1$ nodes where $\lambda(G Ⓩ H) \leq \lambda(G) + \lambda(H)$.*

This theorem was proved in the paper by Reimgold, Vadhan, and Wigderson, in which they presented the zig-zag product. We will prove a slightly simpler version in this lecture, and a further simpler proof can be found in the textbook.

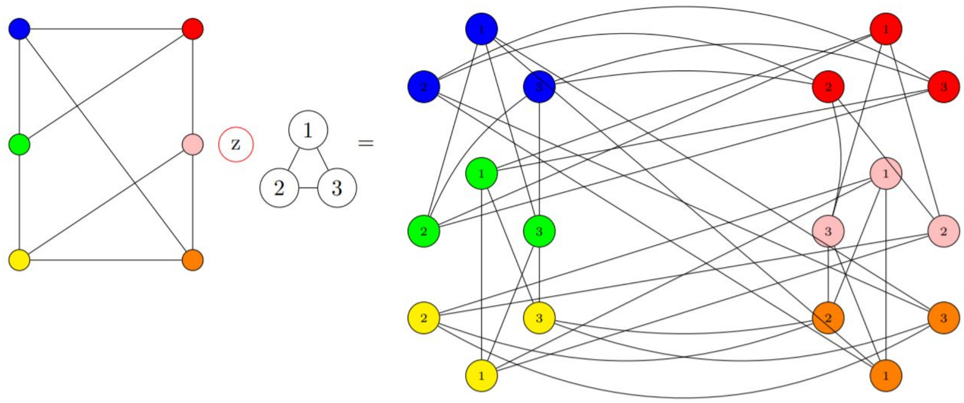

Figure 9.1: An example of zig-zag product.

Lemma 9.4 *$G \textcircled{z} H$ is a D_2^2-regular graph on $N_1 D_1$ nodes where $\lambda(G \textcircled{z} H) \le \lambda(G) + 2\lambda(H)$.*

The proof for this lemma is in the textbook, in which the authors use the Matrix-Decomposition approach.

Lemma 9.5 *$G \textcircled{z} H$ is a D_2^2-regular graph on $N_1 D_1$ nodes where $\lambda(G \textcircled{z} H) \le \lambda(G) + \lambda(H) + \lambda(H)^2$.*

Proof: We use Vector-Decomposition to prove this lemma. Let

$$\widetilde{A_H} = I_{N_1} \otimes A_H$$

be a matrix in $\mathbb{R}^{N_1 D_1 \times N_1 D_1}$ with its diagonal containing several blocks of A_H, corresponding to vertex replacement. Let $P \in \mathbb{R}^{N_1 D_1 \times N_1 D_1}$ be a permutation matrix s.t.

$$P_{(v,i),(u,j)} = 1,$$

iff $e = (u, v) \in E(G)$ where e is the i-th edge incident on v and e is also the j-th edge incident on u. Other entries in P wil be 0.

Claim 9.6 *$A_{G \textcircled{z} H} = \widetilde{A_H} P \widetilde{A_H}$, where $A_{G \textcircled{z} H}$ is the random walk matrix for $G \textcircled{z} H$.*

To prove the above lemma, we should bound the following spectral norm:

$$\max_{x \in \mathbb{R}^{N_1 D_1}, \|x\|_2 = 1, x \perp 1_{N_1 D_1}} \left| x^T \widetilde{A_H} P \widetilde{A_H} x \right|. \tag{9.1}$$

We then use Vector-Decomposition to rewrite

$$x = (x_1, x_2, \ldots, x_{N_1}) = (x_1^{\parallel}, x_2^{\parallel}, \ldots, x_{N_1}^{\parallel}) + (x_1^{\perp}, x_2^{\perp}, \ldots, x_{N_1}^{\perp}) = x^{\parallel} + x^{\perp},$$

where $x_i^{\parallel} \parallel 1_{D_1}, x_i^{\perp} \perp 1_{D_1}$ for all $i \in [N_1]$. Each $x_i^{\parallel}, x_i^{\perp}$ is the vector decomposition restricted on i-th cloud.

We make the following two useful observations: 1) $\widetilde{A_H} x^{\parallel} = x^{\parallel}$ since for each i, $A_H x_i^{\parallel} = x_i^{\parallel}$; 2) $\left\| \widetilde{A_H} x \perp \right\|_2 \le \lambda(H) \left\| x^{\perp} \right\|_2$ since it holds for each i. After substituting $x = x^{\parallel} + x^{\perp}$ into equation 9.1, the remaining thing is to show that:

$$\left| (x^{\parallel} \widetilde{A_H})^T P \widetilde{A_H} x^{\parallel} + 2(x^{\parallel} \widetilde{A_H})^T P \widetilde{A_H} x^{\perp} + (x^{\perp} \widetilde{A_H})^T P \widetilde{A_H} x^{\perp} \right| \le \lambda(G) + \lambda(H) + \lambda(H)^2.$$

We will deal with each of three terms separately and by triangle inequality, we will get our result.

1. $|(x^{\parallel}\widetilde{A_H})^T P \widetilde{A_H} x^{\parallel}| \leq \lambda(G)$.

   We define $y \in \mathbb{R}^{N_1}$ s.t. $y_i = \sqrt{D_1} x^{\parallel}_{i,1}$. If we collapse $x^{\parallel}$ by shrinking each cloud, and normalizing the result, we will get y. We have

$$\begin{aligned}|(x^{\parallel}\widetilde{A_H})^T P \widetilde{A_H} x^{\parallel}| &= |{x^{\parallel}}^T P x^{\parallel}| \quad \text{(by observation 1)}\\ &= |y^T A_G y| \quad \text{(by cancel out 2 } \sqrt{D_1} \text{ factors)}\\ &\leq \lambda(G) \quad \text{(by expansion of } G\text{)}\end{aligned}$$

2. $|2(x^{\parallel}\widetilde{A_H})^T P \widetilde{A_H} x^{\perp}| \leq \lambda(H)$.

$$\begin{aligned}|2(x^{\parallel}\widetilde{A_H})^T P \widetilde{A_H} x^{\perp}| &= |2(x^{\parallel})^T P \widetilde{A_H} x^{\perp}| \quad \text{(by observation 1)}\\ &\leq 2\left\|(x^{\parallel})^T P\right\|_2 \left\|\widetilde{A_H} x^{\perp}\right\|_2 \quad \text{(by Cauchy-Schwartz)}\\ &\leq 2\left\|x^{\parallel}\right\|_2 \lambda(H) \left\|x^{\perp}\right\|_2 \quad \text{(by permuting in the same cloud, and observation 2)}\\ &\leq 2\lambda(H)\frac{\left\|x^{\parallel}\right\|_2^2 + \left\|x^{\perp}\right\|_2^2}{2} \quad \text{(by AM-GM ineq)}\\ &\leq \lambda(H).\end{aligned}$$

3. $|(x^{\perp}\widetilde{A_H})^T P \widetilde{A_H} x^{\perp}| \leq \lambda(H)^2$.

$$\begin{aligned}|(x^{\perp}\widetilde{A_H})^T P \widetilde{A_H} x^{\perp}| &\leq \left\|\widetilde{A_H} x^{\perp}\right\|_2 \|P\|_2 \left\|\widetilde{A_H} x^{\perp}\right\|_2 \quad \text{(by Cauchy-Schwartz)}\\ &\leq \lambda(H)^2 \left\|x^{\perp}\right\|_2^2 \quad \text{(since } \|P\|_2 \text{ is at most 1, and observation 2)}\\ &\leq \lambda(H)^2.\end{aligned}$$

■

9.3 Constructing Explicit Expanders

9.3.1 Mildly Explicit

Let $G_1 = H$ be a $(D^4, D, 1/4)$ expander constructed by brute-force. Let $G_{i+1} = G_i^2 \textcircled{z} G_1$.

Claim 9.7 *G_n is a $(D^{4n}, D^2, 1/2)$ expander.*

Proof: By induction. $G_n^2 : (D^{4n}, D^4, 1/4).G_1 : (D^4, D, 1/4), G_n^2 \textcircled{z} G_1 : (D^{4(n+1)}, D^2, 1/2)$. ■

9.3.2 Strongly Explicit

Let $G_1 = H$ be a $(D^4, D, 1/4)$ expander constructed by brute-force. Let $G_{i+1} = (G_i \otimes G_i)^2 \textcircled{z} G_1$.