The K-armed Dueling Bandits Problem

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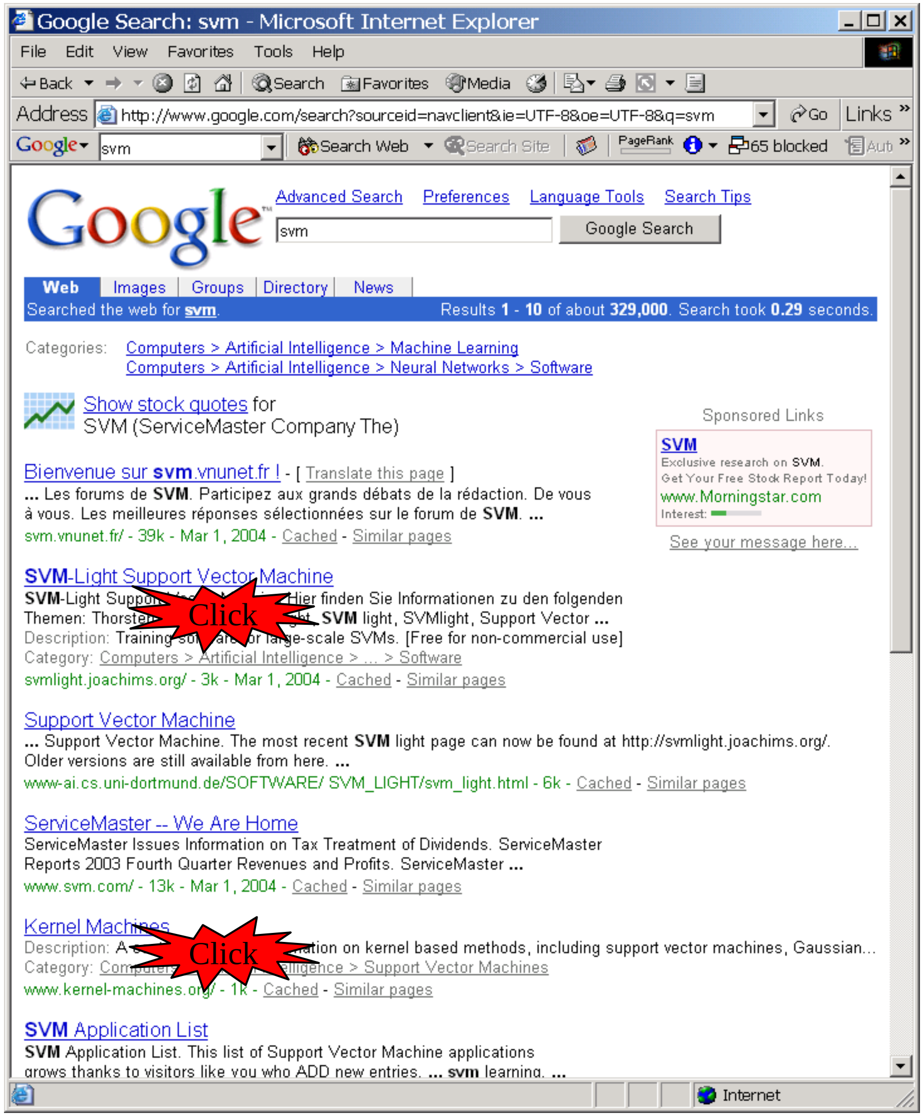
Adaptive Information Systems

- **Retrieval Function: f(q) → r**
  - Input: q (query)
  - Output: r (ranking by relevance)
- **Conventional Systems**
  - One-size-fits-all
  - Hand-tuned and static retrieval function
- **Room for Improvement**
  - Different users need different retrieval functions
  - Different collections need different retrieval functions
- **Machine Learning**
  - Learn improved retrieval functions



Motivation and Outline

- **Setup**
  - Corpus of documents [*known*]
  - Distribution of users and/or queries on corpus [*unknown*]
  - Set of retrieval functions $\{f_1,\ldots,f_K\}$ [*design choice*]
  - Each retrieval function f_i has utility $U(f_i)$ [*unknown*]
- **Question 1: How can one measure utility?**
  - Cardinal vs. ordinal utility measurements
  - Eliciting implicit feedback through interactive experiments
- **Question 2: How to efficiently find f_i with max utility?**
  - Efficiently → minimizing regret + computationally efficient
  - Minimize exposure to suboptimal results during learning
  - Dueling Bandits Problem with efficient algorithm

Approaches to Implicit Utility Elicitation

- **Approach 1: Absolute Metrics (cardinal)**
  - Do metrics derived from observed user behavior provide absolute feedback about retrieval quality of f?
  - For example:
    - U(f) ~ numClicks(f)
    - U(f) ~ 1/abandonment(f)
- **Approach 2: Paired Comparison Tests (ordinal)**
  - Do paired comparison tests provide relative preferences between two retrieval functions f_1 and f_2?
  - For example:
    - $f_1 \succ f_2 \Leftrightarrow \text{pairedCompTest}(f_1, f_2) > 0$

Paired Comparisons: Balanced Interleaving

(u=tj,q="svm")

$f_1(u,q) \rightarrow r_1$

1. Kernel Machines
   http://svm.first.gmd.de/
2. Support Vector Machine
   http://jbolivar.freeservers.com/
3. An Introduction to Support Vector Machines
   http://www.support-vector.net/
4. Archives of SUPPORT-VECTOR-MACHINES ...
   http://www.jiscmail.ac.uk/lists/SUPPORT...
5. SVM-Light Support Vector Machine
   http://ais.gmd.de/~thorsten/svm light/

$f_2(u,q) \rightarrow r_2$

1. Kernel Machines
   http://svm.first.gmd.de/
2. SVM-Light Support Vector Machine
   http://ais.gmd.de/~thorsten/svm light/
3. Support Vector Machine and Kernel ... References
   *http://svm.*research.bell-labs.com/SVMrefs.html
4. Lucent Technologies: SVM demo applet
   *http://svm.*research.bell-labs.com/SVT/SVMsvt.html
5. Royal Holloway Support Vector Machine
   http://svm.dcs.rhbnc.ac.uk

Interleaving(r_1,r_2)

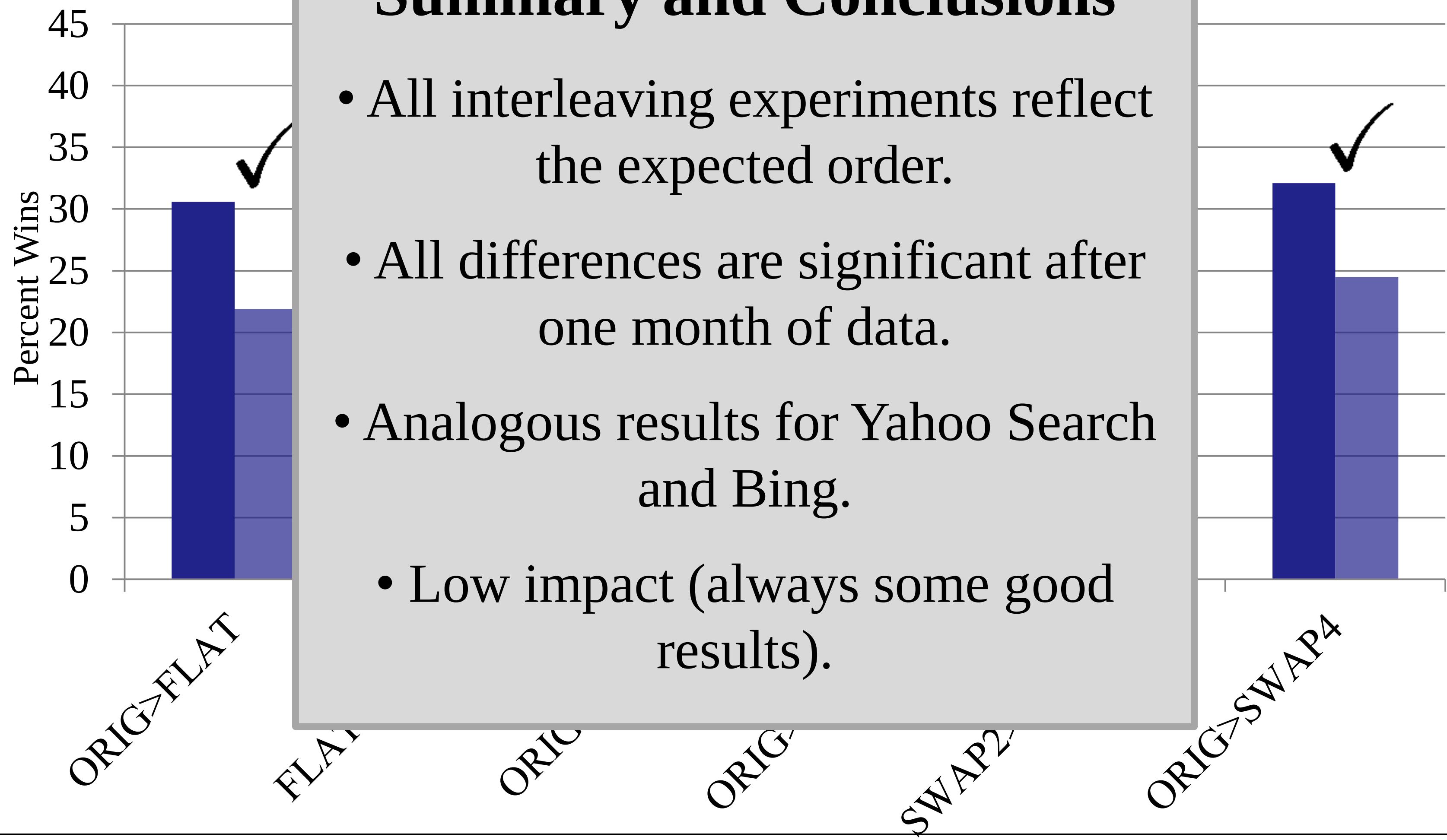
1. Kernel Machines — 1
   http://svm.first.gmd.de/
2. Support Vector Machine — 2
   http://jbolivar.freeservers.com/
3. SVM-Light Support Vector Machine — 2
   http://ais.gmd.de/~thorsten/svm light/
4. An Introduction to Support Vector Machines — 3
   http://www.support-vector.net/
5. Support Vector Machine and Kernel ... References — 3
   *http://svm.*research.bell-labs.com/SVMrefs.html
6. Archives of SUPPORT-VECTOR-MACHINES ... — 4
   http://www.jiscmail.ac.uk/lists/SUPPORT...
7. Lucent Technologies: SVM demo applet — 4
   *http://svm.*research.bell-labs.com/SVT/SVMsvt.html

Invariant:
For all k, top k of balanced interleaving is union of top k_1 of r_1 and top k_2 of r_2 with $k_1=k_2 \pm 1$.

Interpretation: $(r_1 \succ r_2) \leftrightarrow \text{clicks}(\text{topk}(r_1)) > \text{clicks}(\text{topk}(r_2))$

[Joachims/01]

Balanced Interleaving: Results



Paired Comparison Tests: Summary and Conclusions

- All interleaving experiments reflect the expected order.
- All differences are significant after one month of data.
- Analogous results for Yahoo Search and Bing.
- Low impact (always some good results).

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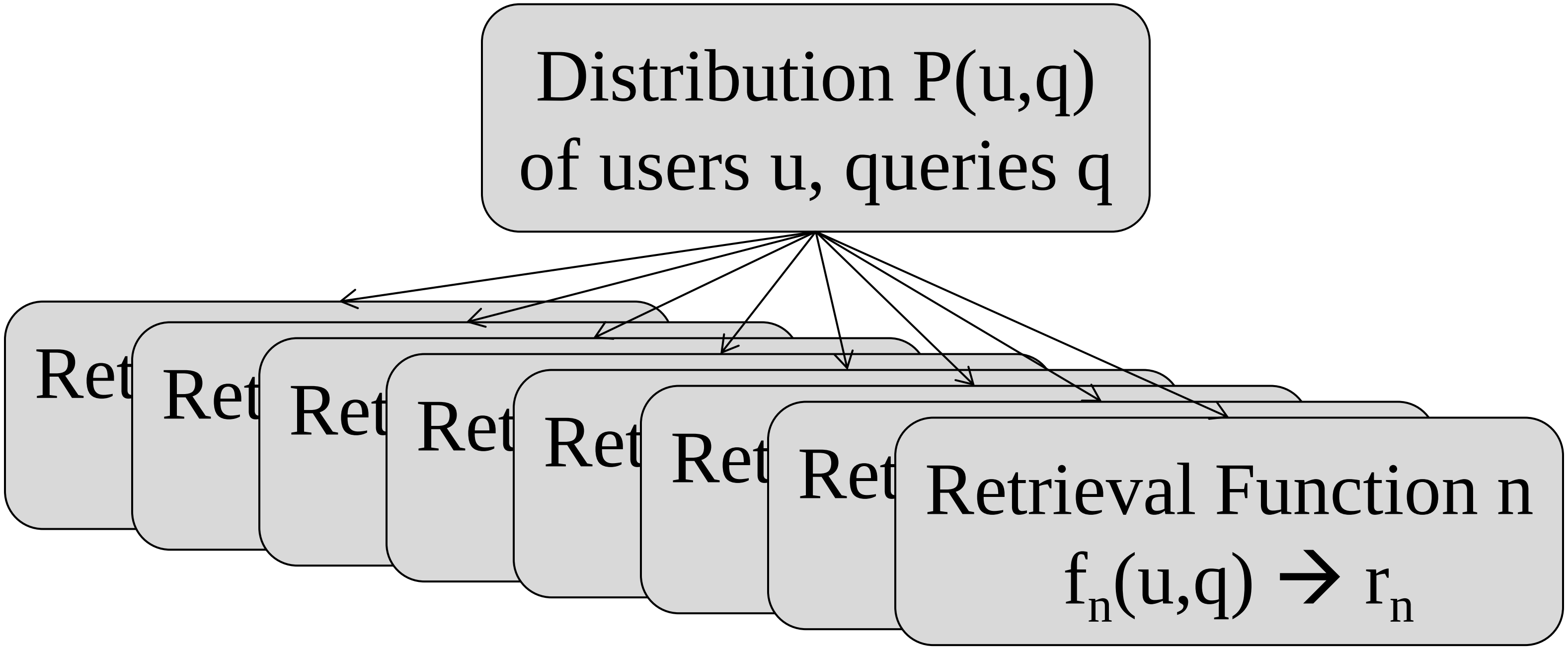
Evaluating Many Retrieval Functions



Task:

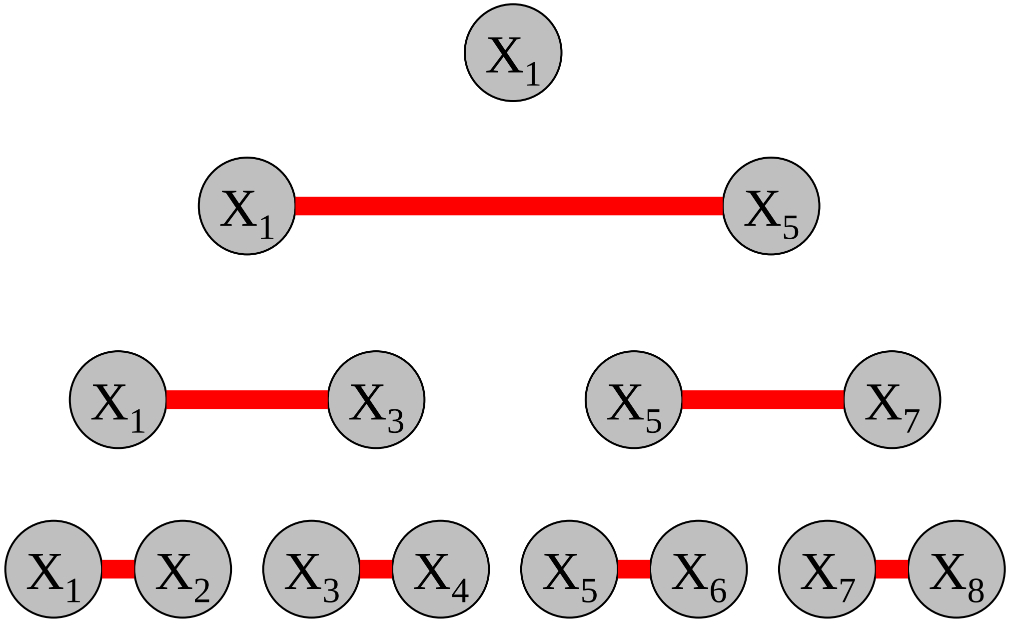
Find f* $\in$ F that gives best retrieval quality over P(u,q)?

Tournament

- **Can you design a tournament that reliably identifies the correct winner?**

  → Noisy Sorting/Max Algorithms:

  - [Feige et al.]: Triangle Tournament Heap $O(n/\varepsilon^2 \log(1/\delta))$ with prob $1-\delta$
  - [Adler et al., Karp & Kleinberg]: optimal under weaker assumptions



Problem: Learning on Operational System

- **Example:**
  - 4 retrieval functions: B > G >> Y > A
  - 10 possible pairs for interactive experiment
    - (B,G) → low cost to user
    - (B,Y) → medium cost to user
    - (Y,A) → high cost to user
    - (B,B) → zero cost to user
    - …
- **Miniming Regret**
  - Algorithm gets to decide on the sequence of pairwise tests
  - Don't present "bad" pairs more often than necessary
  - Trade off (long term) informativeness and (short term) cost

➔ **Dueling Bandits Problem**

Regret for the Dueling Bandits Problem

- **Given:**
  - A finite set H of candidate retrieval functions $f_1 \ldots f_K$
  - A pairwise comparison test $f \succ f'$ on H with $P(f \succ f')$
- **Regret:**
  - $R(A) = \sum_{t=1..T} [P(f^* \succ f_t) + P(f^* \succ f_t') - 1]$
  - f*: best retrieval function in hindsight (assume single f* exists)
  - (f,f'): retrieval functions tested at time t

Example:

Time Step:	t_1	t_2	…	T
Comparison:	$(f_9,f_{12}) \rightarrow f_9$	$(f_5,f_9) \rightarrow f_5$		$(f_1,f_3) \rightarrow f_3$
Regret:	$P(f^*\succ f_9)+P(f^*\succ f_{12})-1$	$P(f^*\succ f_5)+P(f^*\succ f_9)-1$		
	$= 0.9$	$= 0.78$		$= 0.01$

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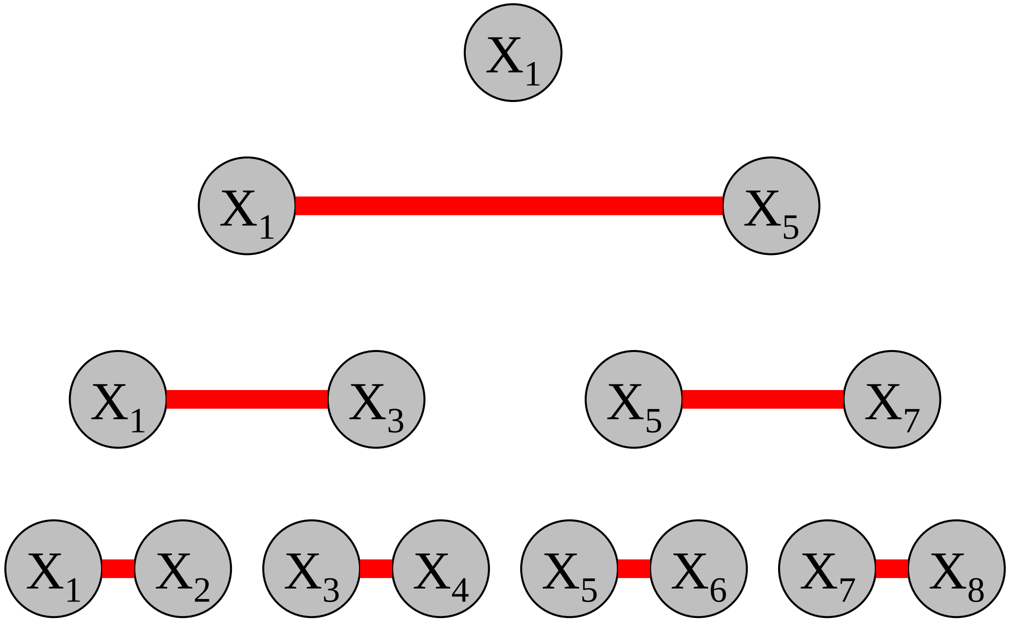
Tournament

- **Can you design a tournament that has low regret?**
  - → Don't know!



Algorithm: Interleaved Filter 1

- **Algorithm**

  InterleavedFilter1(T,W={$f_1 \ldots f_K$})
  - Pick random f' from W
  - $\delta=1/(TK^2)$
  - WHILE |W|>1
    - FOR f ∈ W DO
      - » duel(f',f)
      - » update P_f
    - t=t+1
    - $c_t=(\log(1/\delta)/t)^{0.5}$
    - Remove all f from W with $P_f < 0.5-c_t$ [WORSE WITH PROB 1-δ]
    - IF there exists f'' with $P_{f''} > 0.5+c_t$ [BETTER WITH PROB 1-δ]
      - » Remove f' from W
      - » f'=f''; t=0
  - UNTIL T: duel(f',f')

EXPLORE + EXPLOIT

EXPLOIT

MATCH

ROUND

f_1	f_2	$f'=f_3$	f_4	f_5
0/0	0/0		0/0	0/0

f_1	f_2	$f'=f_3$	f_4	f_5
8/2	7/3		4/6	1/9

f_1	f_2	$f'=f_3$	f_4	
13/2	11/4		7/8	XX

$f'=f_1$	f_2		f_4	
0/0	0/0	XX	0/0	XX

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IF1: Main Result

- **Theorem:** The expected regret of IF1 is

$$E[R_T] = O\left(\frac{K log K}{\epsilon_{1,2}} log T\right)$$

  where ε_{12}=P($f_1 \succ f_2$) – 0.5 and K is the number of bandits.
- **Assumptions:**
  - Strong Stochastic Transitivity: $\varepsilon_{i,k} \geq \max\{\varepsilon_{i,j}, \varepsilon_{j,k}\}$
  - Stochastic Triangle Inequality: $\varepsilon_{i,k} \leq \varepsilon_{i,j} + \varepsilon_{j,k}$
  - ε-winner exists

Assumptions

- **Preference Relation:** $f_i \succ f_j \Leftrightarrow P(f_i \text{ beats } f_j) = 0.5+\varepsilon_{i,j} > 0.5$
- **Weak Stochastic Transitivity:** $f_i \succ f_j$ and $f_j \succ f_k \rightarrow f_i \succ f_k$

$$f_1 \succ f_2 \succ f_3 \succ f_4 \succ f_5 \succ f_6 \succ \ldots \succ f_K$$

- **Strong Stochastic Transitivity:** $\varepsilon_{i,k} \geq \max\{\varepsilon_{i,j}, \varepsilon_{j,k}\}$

$$\varepsilon_{1,4} \geq \varepsilon_{2,4} \geq \varepsilon_{3,4} \geq 0.5 \geq \varepsilon_{5,4} \geq \varepsilon_{6,4} \geq \ldots \geq \varepsilon_{K,4}$$

- **Stochastic Triangle Inequality:** $f_i \succ f_j \succ f_k \rightarrow \varepsilon_{i,k} \leq \varepsilon_{i,j}+\varepsilon_{j,k}$

$$\varepsilon_{1,2} = 0.01 \text{ and } \varepsilon_{2,3} = 0.01 \rightarrow \varepsilon_{1,3} \leq 0.02$$

- **ε-Winner exists:** $\varepsilon = \max_i\{ P(f_1 \text{ beats } f_i)-0.5 \} = \varepsilon_{1,2} > 0$

IF1: Proof Outline

$$E[R_T] \leq \left(1 - \frac{1}{T}\right) E[R_T^{IF1}] + \frac{1}{T} O(T) = O(E[R_T^{IF1}])$$

1. **The probability that IF1 returns suboptimal bandit is less than 1/T**
   a) Probability that a match has wrong winner is at most $\delta = 1/(T\,K^2)$.
   b) Upper bound on the number of matches: K^2
2. **Bound expected regret** $E[R_T^{IF1}]$ **of IF1**
   a) Bound number of duels in a match: $O(1/\varepsilon^2$
   b) Bound regret per match
   c) Bound the number of rounds before IF1 terminates

f_1	…	f_{K-2}	f_{K-1}	$f' = f_K$
0/0	0/0	0/0	0/0	

f_1	…	f_{K-2}	$f' = f_{K-1}$	f_K
0/0	0/0	0/0	0/0	XX

f_1	…	$f' = f_{K-2}$	f_{K-1}	f_K
0/0	0/0	0/0	XX	XX

Lemma 1a: Probability that a Match has Wrong Winner is at most $\delta=1/(T\ K^2)$

- **Proof:**
  - Reminder: Confidence interval $c_t=(\log(1/\delta)/t)^{0.5}$
  - If we declare the wrong winner between f_i and f_j, then observed P_t must have been outside confidence interval.
  - $P(|P_t - E[P_t]| \geq c_t) \leq 2 \exp(-2\ t\ c_t^2) = 2\delta^2 = 2/(T^2\ K^4)$
  - Union bound over all time steps: $2T/(T^2\ K^4) \leq 1/(T\ K^2) = \delta$

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$$E[R_T] \leq \left(1 - \frac{1}{T}\right) E[R_T^{IF1}] + \frac{1}{T} O(T) = O(E[R_T^{IF1}])$$

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Lemma 2a: Bound Number of Duels in a Match

- Consider: match between f_i and f_j with $P(f_i \text{ beats } f_j) = 0.5 + \varepsilon_{i,j}$
- If match is t duels long, then $P_t - c_t \leq 0.5$, otherwise the match would terminate.
- $P(n > t) \leq P(P_t - c_t \leq 0.5) = P(E[P_t] - P_t \geq \varepsilon_{i,j} - c_t)$
- For any $m \geq 4$ and $t=(m \log(TK^2)/\varepsilon^2_{i,j})$, we have $c_t \leq 0.5\ \varepsilon^2_{i,j}$.
- Hoeffding bound → $O(1/\varepsilon^2_{i,j} \log(TK))$ whp

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Lemma 2b: Bound Regret per Match

- **Proof:**
  - Let current incumbant f'=f_j:

f_1	…	f'=f_j	…	f_K
0/0	0/0		0/0	0/0

  - Note: no match involving f_j is longer than $O(1/\varepsilon^2_{1,j} \log(TK))$ whp (Lemma 2a)
  - Each duel(f_i,f_j) incurs ($\varepsilon_{1,j}+\varepsilon_{1,i}$) regret:

f_1 f_i f_j

    - Case $f_i \succ f_j$: Then $\varepsilon_{1,j} + \varepsilon_{1,i} \leq 2\, \varepsilon_{1,j}$ (SST) and regret is bounded $2\varepsilon_{1,j}\, O(1/\varepsilon^2_{1,j} \log(TK)) = O(1/\varepsilon_{1,j} \log(TK)) \leq O(1/\varepsilon_{1,2} \log(TK))$

f_1 f_j f_i

    - Case $f_i \prec f_j$ and $\varepsilon_{j,i} \leq \varepsilon_{1,j}$: Then $\varepsilon_{1,j} + \varepsilon_{1,i} \leq \varepsilon_{1,j} + \varepsilon_{1,j} + \varepsilon_{j,i} \leq 3\, \varepsilon_{1,j}$ due to STI.
$3\varepsilon_{1,j}\, O(1/\varepsilon^2_{1,j} \log(TK)) = O(1/\varepsilon_{1,j} \log(TK)) \leq O(1/\varepsilon_{1,2} \log(TK))$

f_1 f_j f_i

    - Case $f_i \prec f_j$ and $\varepsilon_{j,i} > \varepsilon_{1,j}$: Then $\varepsilon_{1,j} + \varepsilon_{1,i} \leq \varepsilon_{1,j} + \varepsilon_{1,j} + \varepsilon_{j,i} \leq 3\, \varepsilon_{j,i}$ at most $O(1/\varepsilon_{i,j} \log(TK))$ duels.
$3\varepsilon_{j,i}\, O(1/\varepsilon^2_{j,i} \log(TK)) = O(1/\varepsilon_{1,j} \log(TK)) \leq O(1/\varepsilon_{1,2} \log(TK))$

IF1: Proof Outline

$$E[R_T] \leq \left(1 - \frac{1}{T}\right) E[R_T^{IF1}] + \frac{1}{T} O(T) = O(E[R_T^{IF1}])$$

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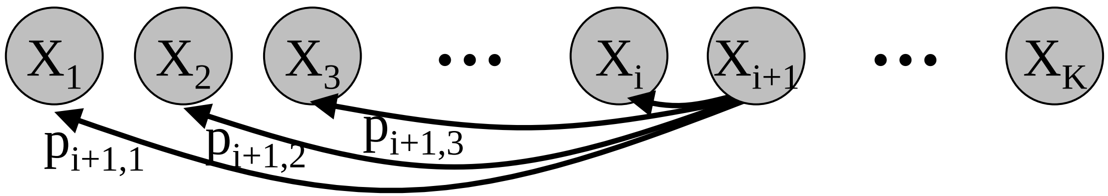
   a) Bound number of duels in a match: $O(1/\varepsilon^2_{i,j} \log(TK))$ whp
   b) Bound regret per match: $O(1/\varepsilon_{1,2} \log(TK))$ whp
   c) Bound the number of rounds before IF1 terminates

Lemma 2c: Bound the Number of Rounds before IF1 Terminates

– Random walk: X_i=1 if f_i becomes incumbant, X_i=0 else

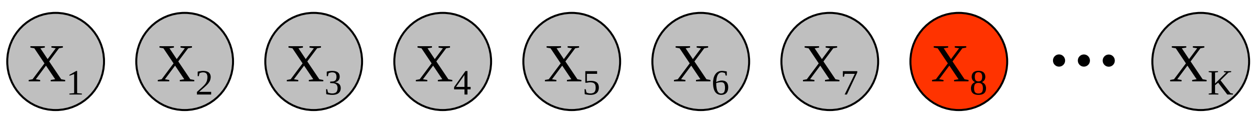


$\rightarrow \sum X_i$ = number of steps in random walk = number of rounds

– Note: If IF1 does not make a mistake, then only forward steps.

– Strong Stochastic Transitivity: $\forall$ i: $p_{i+1,1} \geq p_{i+1,2} \geq \ldots \geq p_{i+1,i}$

$\rightarrow$ Worst case: $p_{i+1,1} = p_{i+1,2} = \ldots = p_{i+1,i} = 1/i$



– $\sum X_i$ = O(log K) rounds whp

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$$E[R_T] \leq \left(1 - \frac{1}{T}\right) E[R_T^{IF1}] + \frac{1}{T} O(T) = O(E[R_T^{IF1}])$$

1. **T**
   l
   a)
   b)

Theorem: IF1 incurs expected regret bounded by

$$E(R_T) \leq O\left(\frac{K \log K}{\epsilon_{1,2}} \log T\right)$$

2. **Bound expected regret** $E[R_T^{IF1}]$ **of IF1**
   a) Bound number of duels in a match: $O(1/\varepsilon^2_{i,j} \log(TK))$ whp
   b) Bound regret per match: $O(1/\varepsilon_{1,2} \log(TK))$ whp
   c) Bound rounds before IF1 terminates: $O(\log K)$ whp

Lower Bound

- **Theorem:** Any algorithm for the dueling bandits problem has regret

$$R_T \leq \Omega\left(\frac{K}{\epsilon_{1,2}} \log T\right)$$

- **Proof:** [Karp/Kleinberg/07][Kleinberg/etal/07]
- **Intuition:**
  - Magically guess the best bandit, just verify guess
  - Worst case: $\forall\, f_i \succ f_j$: $P(f_i \succ f_j)=0.5+\varepsilon$
  - Lemma 2a: Need $O(1/\varepsilon^2 \log T)$ duels to get 1-1/T confidence.

Algorithm: Interleaved Filter 2

- **Algorithm**

  InterleavedFilter1(T,W={$f_1 \ldots f_K$})
  - Pick random f' from W
  - $\delta = 1/(TK^2)$
  - WHILE |W|>1
    - FOR b $\in$ W DO
      - » duel(f',f)
      - » update P_f
    - t=t+1
    - $c_t = (\log(1/\delta)/t)^{0.5}$
    - Remove all f from W with $P_f < 0.5 - c_t$ [WORSE WITH PROB $1-\delta$]
    - IF there exists f'' with $P_{f''} > 0.5 + c_t$ [BETTER WITH PROB $1-\delta$]
      - » Remove f' from W
      - » Remove all f from W that are empirically inferior to f'
      - » f'=f''; t=0
  - UNTIL T: duel(f',f')

f_1	f_2	$f'=f_3$	f_4	f_5
0/0	0/0		0/0	0/0

f_1	f_2	$f'=f_3$	f_4	f_5
8/2	7/3		4/6	1/9

f_1	f_2	$f'=f_3$	f_4	
13/2	11/4		~~7/8~~	XX

$f'=f_1$	f_2		f_4	
0/0	0/0	XX	XX	XX

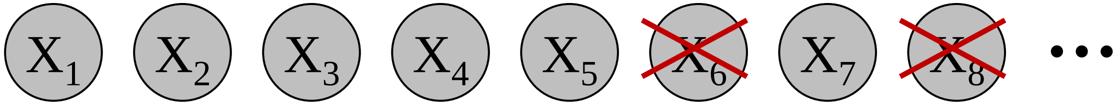
[with Yisong Yue, Josef Broder, Bobby Kleinberg]

Why is it Safe to Remove Empirically Inferior Bandits?

- **Lemma:** Mistakenly pruning a bandit has probability at most $\delta=1/(T K^2)$.
- **Proof:**
  - Mistake: $f_p \succ f_w \succ f_i$ (pruned: f_p, winner: f_w, incumbant: f_i)
  - $B_{n,w,p}$: Given w is winner after n duels, f_p mistakenly pruned.
  - To show: $P(B_{n,w,p}) \leq 1-\delta$ for all n and w.
  - Suppose $P(b_w \succ b_i) = \alpha$ and given $B_{n,w,p}$: $P(b_p \succ b_i) \geq \alpha$.
    $\rightarrow$ $E(S_{w,i}+S_{i,p}) \leq n$.
  - Duels won $S_{w,i} - 0.5n < \mathrm{sqrt}(n \log(1/\delta))$ and $S_{i,p} > 0.5n$
    $\rightarrow$ $S_{w,i} + S_{i,p} - n > \mathrm{sqrt}(n \log(1/\delta))$
  - Hoeffding $P(S_{w,i} + S_{i,p} - n > \mathrm{sqrt}(n \log(1/\delta)) \leq \delta$

Bound the Number of Matches of IF2

- **Lemma:** Assuming IF2 is mistake free, then it plays O(K) matches in expectation.
- **Intuition:**



Regret Bound for IF2

$$E[R_T] \leq \left(1 - \frac{1}{T}\right) E[R_T^{IF1}] + \frac{1}{T} O(T) = O(E[R_T^{IF1}])$$

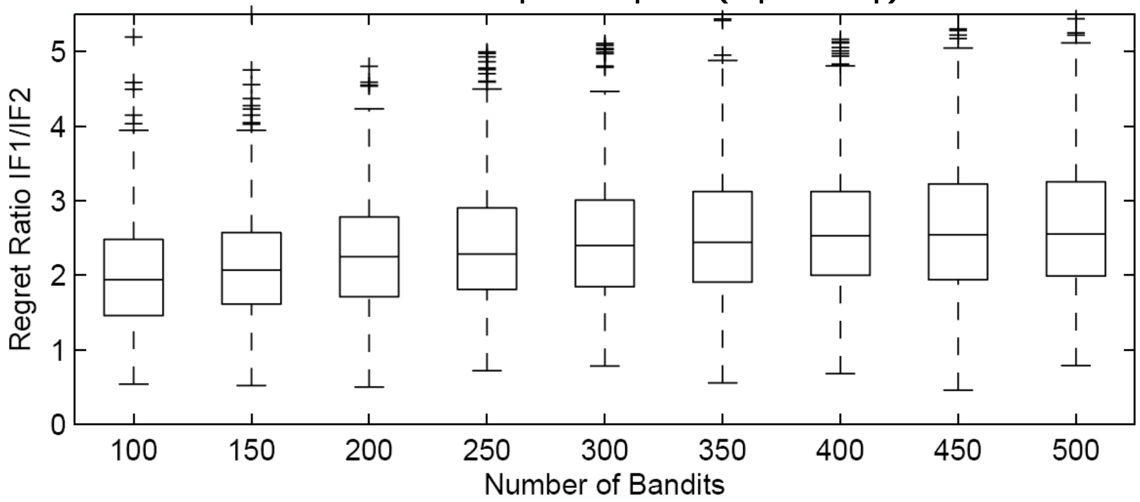
- **Lemma:** Mistakenly pruning a bandit has probability at most $\delta=1/(T K^2)$.
- **Lemma:** Assuming IF2 is mistake free, then it plays O(K) matches in expectation.

Theorem: IF2 incurs expected regret bounded by

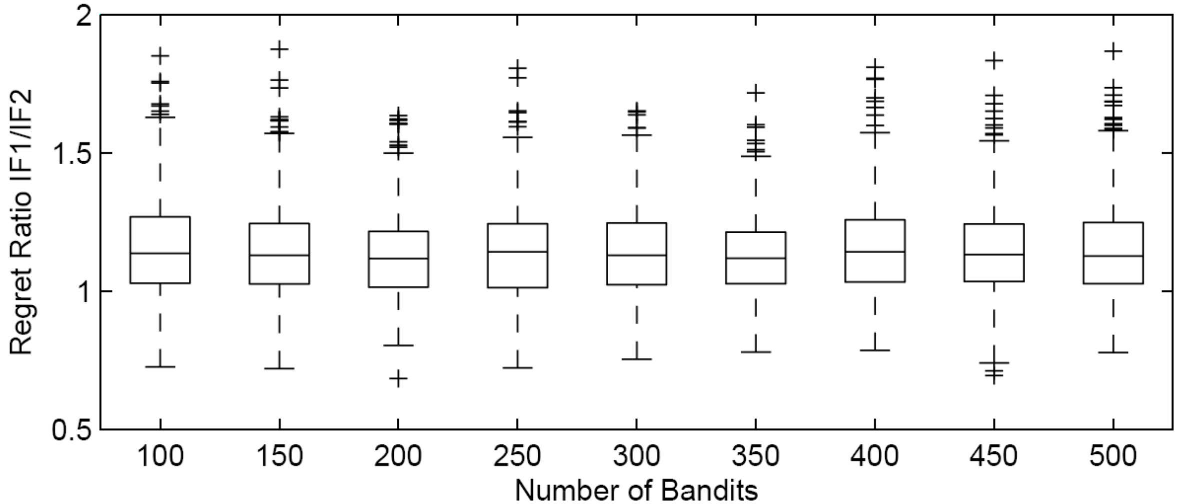
$$E(R_T) \leq O\left(\frac{K}{\epsilon_{1,2}} \log T\right)$$

Experiments: Synthetic Data

- **Lower-Bound data:** $\forall f_i \succ f_i: P(f_i \succ f_i)=0.5+\varepsilon$

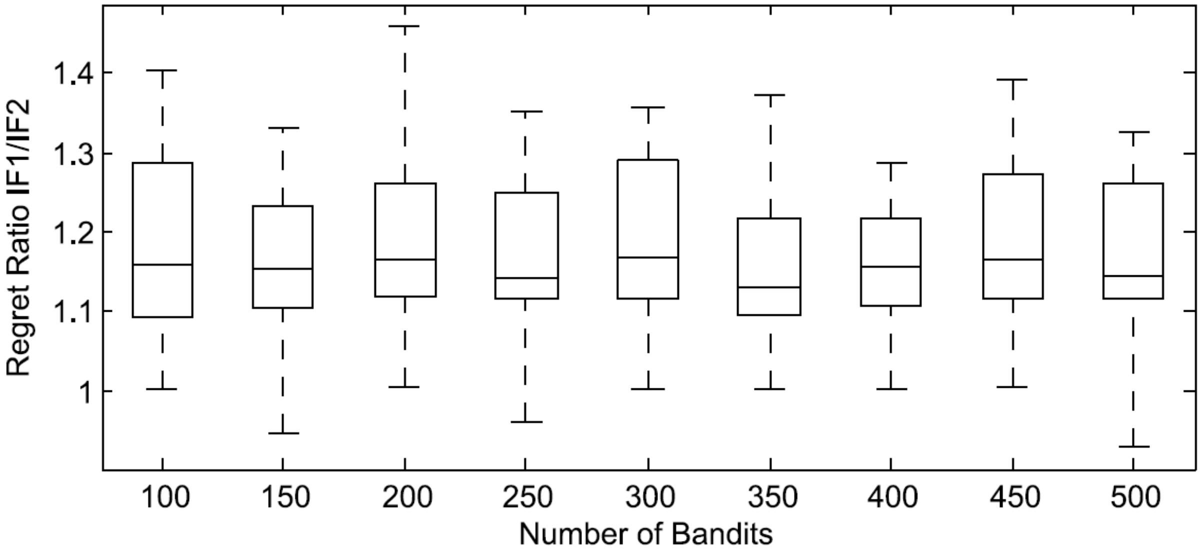


- **Bradley-Terry data**



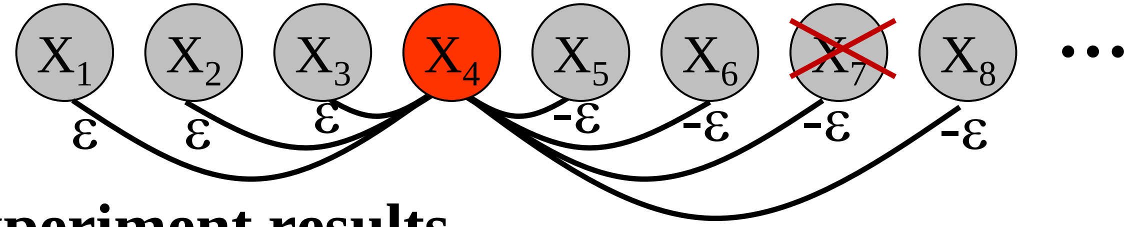
Experiment: Simulated Web Search

- **Microsoft Web Search Data (Chris Burges) with manual relevance assessment**
- **Feedback $f_i \succ f_j$:**
  - Draw query at random
  - Preference $f_i \succ f_j$ (probabilistically) based on NDCG difference of rankings produced by f_i and f_j

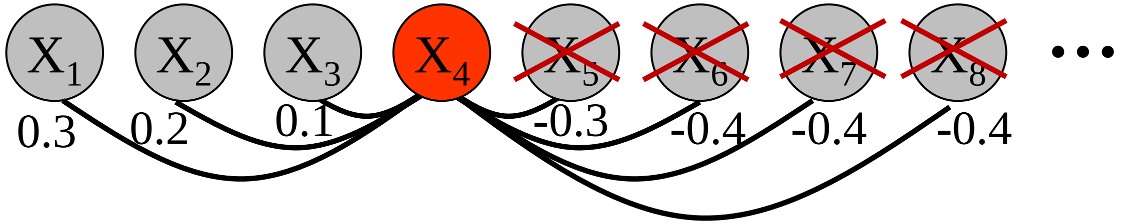


Why not a log-Gap?

- **To achieve log-gap:**
  - Log number of rounds need to be played
  - Most inferior bandits must not get eliminated anyway without pruning.



- **Experiment results**
  - Typically 2-4 rounds largely independent of number of bandits
  - Many bandits much worse, so eliminated before round ends



Summary

- **Dueling Bandits Problem**
  - Only ordinal information about payoffs
  - Algorithms proposes two alternatives, user provides noisy preference.
  - Preference can be interleaving, direct comparison, etc.
- **Interleaved Filter Algorithm**
  - Regret based on win/loss against optimal bandit
  - Strategy: keep incumbent, compare against others, prune inferior
  - O(K/ϵ log T) regret like for bandits with absolute feedback
- **Further Question**
  - Beat-the-Mean-Bandit algorithm for K-armed dueling bandits problem [Yue & Joachims, 2011]
    - Lower variability
    - Relax strong stochastic transitivity
  - Algorithm for finite and convex sets of bandit [Yue & Joachims, 2009]