# Machine Learning Theory (CS 6783)

Lecture 4 : MDL principle, Uniform Rate and Infinite classes

## 1 Recap

1. ERM Algorithm: $\hat{\mathbf{y}}_{\text{ERM}} = \operatorname*{argmin}_{f \in \mathcal{F}} \sum_{(x,y)\in S} \ell(f(x), y)$

2. For the ERM algorithm, we have

$$\mathbb{E}_S \left[ L_D(\hat{\mathbf{y}}_{\text{ERM}}) - \inf_{f \in \mathcal{F}} L_D(f) \right] \leq \mathbb{E}_S \sup_{f \in \mathcal{F}} \left[ \mathbb{E}_{(x,y)\sim D}[\ell(f(x), y)] - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right]$$

3. For finite class,

$$\mathbb{E}_S \sup_{f \in \mathcal{F}} \left[ \mathbb{E}_{(x,y)\sim D}[\ell(f(x), y)] - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right] \leq O\left( \frac{\log(n|\mathcal{F}|)}{n} \right)$$

What about infinite $\mathcal{F}$?

## 2 MDL bound (Occam's Razor Principle)

We saw how one can get bounds for the case when $\mathcal{F}$ has finite cardinality. How about the case when $\mathcal{F}$ has infinite cardinality ? To start with, let us consider the case when $\mathcal{F}$ is a countable set.

**MDL Algorithm:** The MDL learning rule picks the hypothesis in $\mathcal{F}$ as follows :

$$\hat{y}_{\text{mdl}} = \operatorname*{argmin}_{f \in \mathcal{F}} \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) + 3\sqrt{\frac{\log(n/\pi^2(f))}{n}}$$

Interpretation : minimize empirical error while also ensuring that the hypothesis we pick has a large prior $\pi$. Why is this learning rule appealing ?

We will use the below claim to provide us an intuition for why the MDL algorithm is effective.

**Claim 1.** *For any countable set $\mathcal{F}$, any fixed distribution $\pi$ on $\mathcal{F}$,*

$$\mathbb{E}_S \left[ \sup_{f \in \mathcal{F}} \left\{ \left| L_D(f) - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right| - \sqrt{\frac{\log(n/\pi^2(f))}{n}} \right\} \right] \leq \frac{4}{\sqrt{n}}$$

*Proof.* The basic idea is to use Hoeffding bound along with union bound as before, but instead of using same $\epsilon$ for every $f \in \mathcal{F}$ in Hoeffding bound, we use $f$ specific $\epsilon(f)$. We shall specify the exact form of $\epsilon(f)$ later. For now note that, since the losses are bounded by 1,

$$\sup_{f \in \mathcal{F}} \left\{ \left| L_D(f) - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right| - \epsilon(f) \right\} \leq 0 + 2\, \mathbb{1}_{\left\{\sup_{f \in \mathcal{F}} \left\{ \left| L_D(f) - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right| - \epsilon(f) > 0 \right\}\right\}}$$

Hence, taking expectation w.r.t. sample we have that

$$\mathbb{E}_S \left[ \sup_{f \in \mathcal{F}} \left\{ \left| L_D(f) - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right| - \epsilon(f) \right\} \right] \leq 2P \left( \sup_{f \in \mathcal{F}} \left\{ \left| L_D(f) - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right| - \epsilon(f) > 0 \right\} \right)$$

By Hoeffding inequality, for any fixed $f \in \mathcal{F}$

$$P \left( \left| \mathbb{E} \left[ \ell(f(x), y) \right] - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right| - \epsilon(f) > 0 \right) \leq 2 \exp \left( - \frac{\epsilon^2(f) n}{2} \right)$$

Taking union bound we have,

$$P \left( \sup_{f \in \mathcal{F}} \left| \mathbb{E} \left[ \ell(f(x), y) \right] - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right| - \epsilon(f) > 0 \right) \leq \sum_{f \in \mathcal{F}} 2 \exp \left( - \frac{\epsilon^2(f) n}{2} \right)$$

Hence we conclude that

$$\mathbb{E}_S \left[ \sup_{f \in \mathcal{F}} \left\{ \left| L_D(f) - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right| - \epsilon(f) \right\} \right] \leq 4 \sum_{f \in \mathcal{F}} \exp \left( - \frac{\epsilon^2(f) n}{2} \right)$$

For the prior choice of $\pi$ of distribution over set $\mathcal{F}$, let us use

$$\epsilon(f) = \sqrt{\frac{\log(n/\pi^2(f))}{n}}$$

Hence we can conclude that,

$$\begin{aligned} \mathbb{E}_S \left[ \sup_{f \in \mathcal{F}} \left\{ \left| L_D(f) - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right| - \sqrt{\frac{\log(n/\pi^2(f))}{n}} \right\} \right] &\leq 4 \sum_{f \in \mathcal{F}} \exp \left( - \frac{\epsilon^2(f) n}{2} \right) \\ &\leq \frac{4 \sum_f \pi(f)}{\sqrt{n}} = \frac{4}{\sqrt{n}} \end{aligned}$$

□

Let us use the claim above to analyze the learning rule.

**Theorem 2.** *For the MDL algorithm, we have that:*

$$\mathbb{E}_S \left[ L_D(\hat{y}_{\mathrm{mdl}}) \right] \leq \inf_{f \in \mathcal{F}} \left\{ L_D(f) + \sqrt{\frac{\log(n/\pi^2(f))}{n}} \right\} + \frac{4}{\sqrt{n}}$$

*Proof.* Note that from the Claim 1, we have that,

$$\mathbb{E}_S\left[L_D(\hat{y}_{\mathrm{mdl}}) - \frac{1}{n}\sum_{t=1}^{n}\ell(\hat{y}_{\mathrm{mdl}}(x_t), y_t) - \sqrt{\frac{\log(n/\pi^2(\hat{y}_{\mathrm{mdl}}))}{n}}\right] \leq \frac{4}{\sqrt{n}}$$

By definition of $\hat{y}_{\mathrm{mdl}}$ we can conclude that

$$\mathbb{E}_S\left[L_D(\hat{y}_{\mathrm{mdl}}) - \inf_{f\in\mathcal{F}}\left\{\frac{1}{n}\sum_{t=1}^{n}\ell(\hat{y}_{\mathrm{mdl}}(x_t), y_t) + \sqrt{\frac{\log(n/\pi^2(\hat{y}_{\mathrm{mdl}}))}{n}}\right\}\right] \leq \frac{4}{\sqrt{n}}$$

In other words,

$$\mathbb{E}_S\left[L_D(\hat{y}_{\mathrm{mdl}})\right] \leq \mathbb{E}_S\left[\inf_{f\in\mathcal{F}}\left\{\frac{1}{n}\sum_{t=1}^{n}\ell(f(x_t), y_t) + \sqrt{\frac{\log(n/\pi^2(f))}{n}}\right\}\right] + \frac{4}{\sqrt{n}}$$

Let $f_D = \operatorname*{argmin}_{f\in\mathcal{F}} L_D(f) + \sqrt{\frac{\log(n/\pi^2(f))}{n}}$, replacing the infimum above we conclude that

$$\begin{aligned}\mathbb{E}_S\left[L_D(\hat{y}_{\mathrm{mdl}})\right] &\leq \mathbb{E}_S\left[\frac{1}{n}\sum_{t=1}^{n}\ell(f_D(x_t), y_t) + \sqrt{\frac{\log(n/\pi^2(f_D))}{n}}\right] + \frac{4}{\sqrt{n}} \\ &= L_D(f_D) + \sqrt{\frac{\log(n/\pi^2(f_D))}{n}} + \frac{4}{\sqrt{n}} \\ &= \inf_{f\in\mathcal{F}}\left\{L_D(f) + \sqrt{\frac{\log(n/\pi^2(f))}{n}}\right\} + \frac{4}{\sqrt{n}} \qquad (1) \\ & \qquad (2)\end{aligned}$$

□

Thus with the above bound, even for countably infinite $\mathcal{F}$ we can get bounds on $\mathbb{E}_S\left[L_D(\hat{y})\right] - \min_{f\in\mathcal{F}} L_D(f)$ as

$$\mathbb{E}_S\left[L_D(\hat{y})\right] - \min_{f\in\mathcal{F}} L_D \leq \sqrt{\frac{\log(n/\pi^2(f_D))}{n}} + \frac{4}{\sqrt{n}}$$

that decreases with $n$, however the rate depends on $\log(1/\pi(f_D))$ where $f_D = \operatorname*{argmin}_{f\in\mathcal{F}} L_D(f)$.

## 3 Infinite Hypothesis Class : first attempt

As a first attempt, one can think of approximating the function class to desired accuracy by a finite number of representative elements. We call this a point-wise cover.

**Definition 1.** *We say that set $\mathcal{F}_\delta = \{\tilde{f}_1, \ldots, \tilde{f}_N\}$ is an $\delta$ point-wise cover for function class $\mathcal{F}$ if $\forall f \in \mathcal{F}$ there exists $i \in [N]$ s.t.*

$$\sup_{x,y} |\ell(f(x), y) - \ell(\tilde{f}_i(x), y)| \leq \delta$$

*Further define $N(\delta)$ to be the smallest $N$ such that there exists an $\delta$ cover of $\mathcal{F}$ of cardinality at most $N$.*

**Claim 3.** *For any function class $\mathcal{F}$, we have that*

$$\mathcal{V}_n^{\text{stat}}(\mathcal{F}) \leq \inf_{\delta > 0} \left\{ 4\delta + \sqrt{\frac{\log N(\delta)}{n}} \right\}$$

*Proof.* Let $\mathcal{F}_\delta = \{\tilde{f}_1, \ldots, \tilde{f}_{N(\delta)}\}$ be an $\delta$ cover for the function class $\mathcal{F}$. Further for every $f \in \mathcal{F}$, let $i(f)$ correspond to the index of the element in $\mathcal{F}_\delta$ that is $\delta$ close to that $f$. Now note that,

$$\begin{aligned}
&\mathbb{E}_S \left[ \sup_{f \in \mathcal{F}} \left\{ \mathbb{E}\left[\ell(f(x), y)\right] - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) \right\} \right] \\
&\qquad \leq \mathbb{E}_S \left[ \max_{i \in [N_\delta]} \left\{ \mathbb{E}\left[\ell(\tilde{f}_i(x), y)\right] - \frac{1}{n} \sum_{t=1}^{n} \ell(\tilde{f}_i(x_t), y_t) \right\} \right] \\
&\qquad\quad + \mathbb{E}_S \left[ \sup_{f \in \mathcal{F}} \left| \mathbb{E}\left[\ell(f(x), y)\right] - \frac{1}{n} \sum_{t=1}^{n} \ell(f(x_t), y_t) - \mathbb{E}\left[\ell(\tilde{f}_{i(f)}(x), y)\right] + \frac{1}{n} \sum_{t=1}^{n} \ell(\tilde{f}_{i(f)}(x_t), y_t) \right| \right] \\
&\qquad \leq \sqrt{\frac{\log N(\delta)}{n}} + 4\delta
\end{aligned}$$

where the first term in the last inequality is by using the finite class bound and the second term is by using the definition of $\delta$ cover as $\tilde{f}_{i(f)}$ is $\delta$ close to $f$. Since choice of $\delta$ was arbitrary we can take the infimum over choices of $\delta$ to conclude the proof. $\square$

**Example : linear predictor, absolute loss, 1 dimension**
$f(x) = f \cdot x, \quad \mathcal{F} = \mathcal{X} = [-1, 1], \quad \mathcal{Y} = [-1, 1], \quad \ell(y', y) = |y - y'|$

$N_\delta = \frac{2}{\delta}$, Cover given by $f_1 = -1, f_2 = -1 + \delta, \ldots, f_{N_\delta - 1} = 1 - \delta, f_{N_\delta} = 1$.

$$V_n^{\text{stat}}(\mathcal{F}) \leq \sqrt{\frac{\log n}{n}}$$

**Example : linear predictor/loss, $d$ dimensions**
$f(x) = \mathbf{f}^\top \mathbf{x}$. $\mathcal{F} = \mathcal{X} = \{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\|_2 \leq 1\}$. $\mathcal{Y} = [-1, 1]$. $\ell(y', y) = y \cdot y'$

$N_\delta = \Theta\left(\frac{2}{\delta}\right)^d$

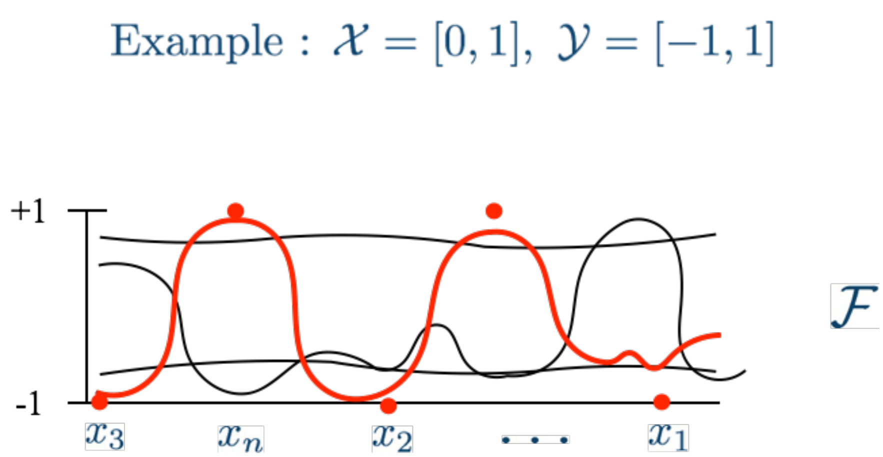
$$V_n^{\text{stat}}(\mathcal{F}) \leq \sqrt{\frac{d \log n}{n}}$$

**Example : thresholds**
$f(x) = \text{sign}(f - x), \quad \mathcal{F} = \mathcal{X} = [-1, 1], \quad \mathcal{Y} = \{-1, 1\}, \quad \ell(y', y) = \mathbb{1}_{\{y \neq y'\}}, \quad N_\delta = \infty$ for any $\delta < 1$.

## 4 Symmetrization and Rademacher Complexity

$$
\begin{aligned}
&\mathbb{E}_S\left[L_D(\hat{y}_{\text{erm}})\right] - \inf_{f\in\mathcal{F}} L_D(f) \\
&\qquad \le \mathbb{E}_S\left[\sup_{f\in\mathcal{F}}\left\{\mathbb{E}\left[\ell(f(x),y)\right] - \frac{1}{n}\sum_{t=1}^{n}\ell(f(x_t),y_t)\right\}\right] \\
&\qquad \le \mathbb{E}_{S,S'}\left[\sup_{f\in\mathcal{F}}\left\{\frac{1}{n}\sum_{t=1}^{n}\ell(f(x'_t),y'_t) - \frac{1}{n}\sum_{t=1}^{n}\ell(f(x_t),y_t)\right\}\right] \\
&\qquad = \mathbb{E}_{S,S'}\mathbb{E}_{\epsilon}\left[\sup_{f\in\mathcal{F}}\left\{\frac{1}{n}\sum_{t=1}^{n}\epsilon_t(\ell(f(x'_t),y'_t) - \ell(f(x_t),y_t))\right\}\right] \\
&\qquad \le 2\mathbb{E}_S\mathbb{E}_{\epsilon}\left[\sup_{f\in\mathcal{F}}\left\{\frac{1}{n}\sum_{t=1}^{n}\epsilon_t\ell(f(x_t),y_t)\right\}\right] \\
&\qquad =: \mathcal{R}_n(\ell\circ\mathcal{F})
\end{aligned}
$$

Where in the above each $\epsilon_t$ is a Rademacher random variable that is $+1$ with probability $1/2$ and $-1$ with probability $1/2$. The above is called Rademacher complexity of the loss class $\ell\circ\mathcal{F}$. In general Rademacher complexity of a function class measures how well the function class correlates with random signs. The more it can correlate with random signs the more complex the class is.




How does this help?