Time Varying LQR

$$x_{t+1} = A_t x_t + B_t u_t$$

$$c(x_t, u_t) = x_t^T Q_t x_t + u_t^T R_t u_t$$

$$K_t = \left(- B_t^T V_{t+1} B_t + R_t \right)^{-1} B_t^T V_{t+1} A_t$$

Affine LQR

$$x_{t+1} = A_t x_t + B_t u_t + \alpha_t^{off}$$

Homogenous Coordinate.

$$\tilde{x}_t = \begin{bmatrix} x_t \\ 1 \end{bmatrix}$$ "Augmented state"

$$\underbrace{\begin{bmatrix} x_{t+1} \\ 1 \end{bmatrix}}_{\tilde{x}_{t+1}} = \underbrace{\begin{bmatrix} A_t & \alpha_t^{off} \\ 0 & 1 \end{bmatrix}}_{\tilde{A}_t} \underbrace{\begin{bmatrix} x_t \\ 1 \end{bmatrix}}_{\tilde{x}_t} + \underbrace{\begin{bmatrix} B_t \\ 0 \end{bmatrix}}_{\tilde{B}_t} u_t$$

$$c(x_t, u_t) = x_t^T Q x_t + u_t^T R u_t + G^T x_t + C_{off}$$

Cost:

$$= \underbrace{[x_t \quad 1]}_{\tilde{x}_t} \underbrace{\begin{bmatrix} Q & \frac{1}{2} G^T \\ \frac{1}{2} G & C_{off} \end{bmatrix}}_{\tilde{Q}_t} \underbrace{\begin{bmatrix} x_t \\ 1 \end{bmatrix}}_{\tilde{x}_t} + u_t^T R_t u_t$$

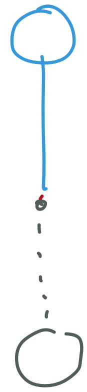
$$\tilde{K}_t \leftarrow \text{TV-LQR}\left(\tilde{A}_t,\ \tilde{B}_t,\ \tilde{Q}_t,\ R_t\right)$$

$$u_t = \tilde{K}_t^{\,M\times(N+1)}\ \tilde{x}_t^{\,(N+1)\times 1}$$

$$= \begin{array}{cc} \scriptstyle M\times N & \scriptstyle M\times 1 \end{array}$$

$$= \left[\, K_t \;\vdots\; u_t^{off} \,\right] \begin{bmatrix} x_t \\ 1 \end{bmatrix} = K_t x_t + u_t^{off}$$

ILQR



$x_{t+1} = f(x_t, u_t)$
DYNAMICS

$C(x_t, u_t)$
COST

INIT: START WITH AN INTIAL GUESS

$$\bar{u}_{0:T-1} = \{\bar{u}_0,\ \bar{u}_1,\ - - - -\ \bar{u}_{T-1}\}$$

STEP 1: FORWARD PASS.

$$\bar{U}_{0:T-1} \rightarrow f \rightarrow \bar{X}_{0:T-1}$$

$$\begin{array}{ccccccccc} \bar{x}_0 & \xrightarrow{f} & \bar{x}_1 & \xrightarrow{f} & \bar{x}_2 & \rightarrow & - - \bar{X}_t - - & & \bar{x}_{T-1} \\ \bar{u}_0 & & \bar{u}_1 & & \vec{u}_2 & & \bar{u}_t & & \bar{u}_{T-1} \end{array}$$

STEP 2: LINEARIZE DYNAMIZE ABOUT FORWARD PASS

$$x_{t+1} = f(x_t, u_t)$$

$$f(x_t, u_t) = f(\bar{x}_t, \bar{u}_t) + \overset{A_t}{\left.\frac{\partial f}{\partial x}\right|_{\bar{x}_t}} (x_t - \bar{x}_t)$$

$$+ \left.\frac{\partial f}{\partial u}\right| (u_t - \bar{u}_t)$$

$\bar{u}_t$

B_t

$$x_{t+1} = f(\bar{x}_t, \bar{u}_t) + A_t(x_t - \bar{x}_t) + B_t(u_t - \bar{u}_t)$$

$$= A_t x_t + B_t u_t + \underbrace{\left(f(\bar{x}_t, \bar{u}_t) - A_t \bar{x}_t - B_t \bar{u}_t \right)}_{\alpha_t^{off}}$$

Quadricize the cost $c(x_t, u_t)$

$$c(x_t, u_t) = c(\bar{x}_t, \bar{u}_t) + \underbrace{\frac{\partial c}{\partial x}}_{G_t}\Big|_{\bar{x}_t, \bar{u}_t} (x_t - \bar{x}_t)$$

$$+ \frac{1}{2}(x_t - \bar{x}_t)^T \underbrace{\frac{\partial^2 c}{\partial x^2}}_{Q_t}\Big|_{\bar{x}_t, \bar{u}_t} (x_t - \bar{x}_t)$$

$$+ u_t^T R u_t$$

$$= x_t^T Q_t x_t + \underbrace{(G_t - \bar{x}_t Q_t)}_{\text{Linear}} x_t + \underbrace{\left[c(\bar{x}_t, \bar{u}_t) + \frac{1}{2} \bar{x}_t Q_t \bar{x}_t \right]}_{C_{off}}$$

$$+ u_t^T R u_t$$

$$= \begin{bmatrix} x_t & 1 \end{bmatrix} \tilde{Q}_t \begin{bmatrix} x_t \\ 1 \end{bmatrix} + u_t^T R_t u_t$$

Step3: Call Affine LQR.

$$\tilde{K}_t.$$

STEP 4: COMPUTE NEW CONTROL TRAJECTORY

$$x_0 \longrightarrow \tilde{K}_0 \longrightarrow u_0 \longrightarrow x_1 = f(x_0, u_0)$$

$$x_1 \xrightarrow{\tilde{K}_1} u_1$$

$$u_1 \longrightarrow x_2 = f(x_1, u_1) \longrightarrow \cdots$$