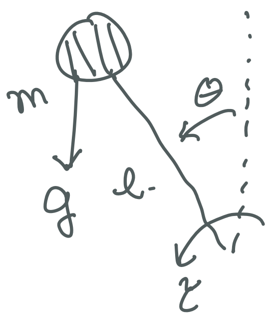
① WRITE OUT MDP FOR INV PENDULUM

② ANALYTICALLY DERIVE OPTIMAL VALUE (VALUE ITERATION)



$$m l^2 \ddot{\theta} = \tau + mgl \sin\theta$$

$$\ddot{\theta} = \frac{g}{l} \sin\theta + \frac{\tau}{ml^2}$$

θ is small
θ ≲ 30°

$$\boxed{\ddot{\theta} = \frac{g}{l}\theta + \frac{\tau}{ml^2}}$$

STATE

θ

$\dot{\theta}$

GOAL: ANALYTICALLY DERIVE VALUE

$$V^*(x_t) = \min_{u_t} \left[c(x_t, u_t) + V^*(x_{t+1}) \right]$$

What's the optimal value at T-1 ?

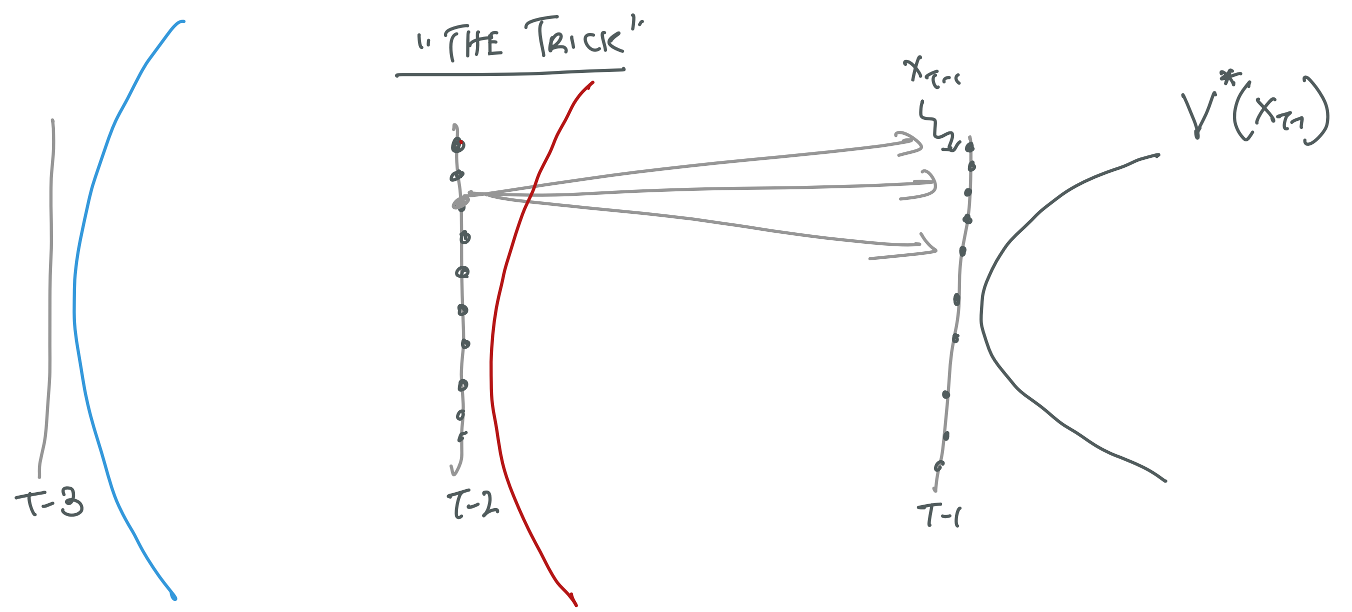
$$V^*(x_{T-1}) = \min_{u_{T-1}} \left[c(x_{T-1}, u_{T-1}) + 0 \right]$$

$$= \min_{u_{T-1}} \left[x_{T-1}^T Q x_{T-1} + u_{T-1}^T R u_{T-1} \right]$$

$$\frac{\partial}{\partial u_{T-1}} (\cdot) = 0 \Rightarrow 2 u_{T-1}^T R = 0 \Rightarrow u_{T-1} = 0$$

$$V^*(x_{T-1}) = x_{T-1}^T Q x_{T-1}$$

QUADRATIC



Assume that $V^*(x_{t+1})$ is a quadratic is this symmetric?

$$V^*(x_{t+1}) = x_{t+1}^T V_{t+1} x_{t+1}$$

$$V^*(x_t) = \min_{u_t} \left[c(x_t, u_t) + V^*(x_{t+1}) \right]$$

$$= \min_{u_t} \left[x_t^T Q x_t + u_t^T R u_t + x_{t+1}^T V_{t+1} x_{t+1} \right]$$

$$Ax_t + Bu_t$$

$$= \min_{u_t} \left[x_t^T Q x_t + u_t^T R u_t + (Ax_t + Bu_t)^T V_{t+1} (Ax_t + Bu_t) \right]$$

$$\frac{\partial}{\partial u_t}(\cdot) = 0$$

$$u_t^T R + u_t^T R^T$$

$$u_t^T (R + R^T) = 2u_t^T R$$

$$\frac{\partial}{\partial x}(Ax) = A$$

$$0 + 2u_t^T R + 2(Ax_t + Bu_t)^T V_{t+1} B$$

$$= 0$$

$$\cancel{2} u_t^T R + \cancel{2} (Ax_t + Bu_t)^T V_{t+1} B = 0$$

(Take a transpose)

$$R u_t + B^T V_{t+1} (Ax_t + Bu_t) = 0$$

$$(R + B^T V_{t+1} B) u_t = - B^T V_{t+1} A x_t$$

$$u_t = \boxed{- (R + B^T V_{t+1} B)^{-1} B^T V_{t+1} A x_t}$$

K_t

$$= K_t \, x_t$$

$$V^*(x_t) = x_t^T Q x_t + u_t^T R u_t + (Ax_t + Bu_t)^T V_{t+1} (Ax_t + Bu_t)]$$

$$= x_t^T Q x_t + x_t^T K_t^T R K_t x_t + (Ax_t + BK_t x_t)^T V_{t+1} (Ax_t + BK_t x_t)$$

$$x_t^T Q x_t + x_t^T K_t^T R K_t x_t + x_t^T (A + BK_t)^T V_{t+1} (A+BK_t) x_t$$

$$= x_t^T \Big(\underbrace{Q + K_t^T R K_t + (A+BK_t)^T V_{t+1} (A+BK_t)}_{V_t} \Big) x_t$$