SWING UP OF A PENDULUM

Time Varying LQR

$$x_{t+1} = A_t x_t + B_t u_t$$

$$c(x_t, u_t) = x_t^T Q_t x_t + u_t^T R_t u_t$$

$$K_t = \left(-B_t^T V_{t+1} B_t + R_t\right)^{-1} B_t^T V_{t+1} A_t$$

Affine LQR

$$x_{t+1} = A_t x_t + B_t u_t + x_t^{off}$$

$$\underbrace{\begin{bmatrix} x_{t+1} \\ 1 \end{bmatrix}}_{\tilde{x}_{t+1}} = \underbrace{\begin{bmatrix} A_t & x_t^{off} \\ 0 & 1 \end{bmatrix}}_{\tilde{A}_t} \underbrace{\begin{bmatrix} x_t \\ 1 \end{bmatrix}}_{\tilde{x}_t} + \underbrace{\begin{bmatrix} B_t \\ 0 \end{bmatrix}}_{\tilde{B}_t} u_t$$

$$\tilde{x}_{t+1} = \tilde{A}_t \tilde{x}_t + \tilde{B}_t u_t$$

Similarly

$$c(x_t, u_t) = x_t^T Q_t x_t + u_t^T R_t u_t + G^T x_t + c_{off}$$

$$= \tilde{x}_t^T \underbrace{\begin{bmatrix} \overset{N\times N}{Q_t} & \overset{N\times 1}{\frac{1}{2}G} \\ \underset{1\times N}{\frac{1}{2}G^T} & \underset{1\times 1}{c_{off}} \end{bmatrix}}_{\tilde{Q}_t \; (N+1),(N+1)} \tilde{x}_t + \ldots$$

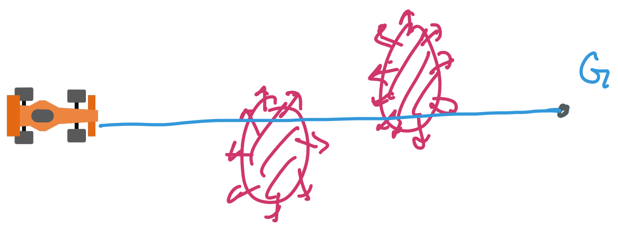
$$[\,x_t^T \quad 1\,]$$

$$\tilde{K}_t = \text{TVLQR}(\hat{A}_t, \hat{B}_t, \hat{Q}_t, \hat{R}_t)$$

$$\underset{M\times 1}{u_t} = \underset{M\times(N+1)}{\tilde{K}_t}\ \underset{(N+1)\times 1}{\tilde{x}_t} = \left[\ \underset{M\times N}{K_t}\ \vdots\ u_t^{off}\ \right]\begin{bmatrix} x_t \\ \hline 1 \end{bmatrix}$$

$$= K_t x_t + u_t^{off}$$

ILQR



Dynamics

$$x_{t+1} = f(x_t, u_t)$$

Cost

$$c(x_t, u_t) = \omega_1 \|x_t - x_G\|^2 + \omega_2 \|u\|^2 + \omega_3 \cdot C_{obs}(x_t)$$

non-linear, non-convex

Init: Start with some initial guess

Step 1: Forward Pass: Rollout

$$\bar{u}_{0:t-1} \rightsquigarrow f \rightsquigarrow \bar{x}_{0:t-1}$$

Step 2: Linearize Dynamics about $\boxed{(\bar{x}_{0:t-1}, \bar{u}_{0:t-1})}$

$$x_{t+1} = f(x_t, u_t)$$

$$\underbrace{(x_{t+1} - \bar{x}_{t+1})}_{\delta x_{t+1}} = \underbrace{f(x_t, u_t)\Big|_{\bar{x}_t, \bar{u}_t} - \bar{x}_{t+1}}_{\delta x_t^{off}} + \underbrace{\frac{\partial f}{\partial x}\Big|_{\bar{x}_t}}_{A_t} \overbrace{\boxed{(x_t - \bar{x}_t)}}^{\delta x_t} + \frac{\partial f}{\partial u}\Big|_{\bar{u}_t} \overbrace{\boxed{(u_t - \bar{u}_t)}}^{\delta u_t}$$

"constant"

$$\delta x_{t+1} = A_t\, \delta x_t + \overset{B_t}{B_t\, \delta u_t} + \delta x^{off}_t$$

Step 3 Quadricize Cost about $\bar{x}_{0:t}, \bar{u}_{0:t-1}$

$$c(x_t, u_t) = \underbrace{c(x_t, u_t)\Big|_{\bar{x}_t, \bar{u}_t}}_{\text{"constant"}} + \frac{\partial c}{\partial x}\Big| \underbrace{(x_t - \bar{x}_t)}_{\delta x_t} +$$

"c^{off}"

$$+ \frac{1}{2}(x_t - \bar{x}_t)^T \frac{\partial^2 c}{\partial x^2} \underbrace{(x_t - \bar{x}_t)}_{\delta x_t}$$

$$+ \underbrace{\cancel{\frac{\partial c}{\partial u}}}_{=0}\Big| (u_t - \bar{u}_t) + \frac{1}{2}(u_t - \bar{u}_t)^T \frac{\partial^2 c}{\partial u^2}(u_t - \bar{u}_t) + \dots$$

$$= \begin{bmatrix} \delta x_t^T & 1 \end{bmatrix} \underbrace{\begin{bmatrix} \frac{1}{2}\frac{\partial^2 c}{\partial x^2} & \frac{1}{2}\frac{\partial c}{\partial x} \\ \frac{1}{2}\frac{\partial c}{\partial x} & c^{off} \end{bmatrix}}_{\tilde{Q}_t} \begin{bmatrix} \delta x_t \\ 1 \end{bmatrix} +$$

Step 4: Call Affine LQR [Backward Pass]

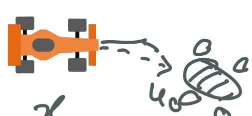
$$\tilde{K}_t = (\tilde{A}_t, \tilde{B}_t, \tilde{Q}_t, \tilde{R}_t)$$

Step 5: New Forward Pass

$t = 0$ $\quad \tilde{K}_0$ $\quad \delta u_0 = \tilde{K}_0\, \delta \tilde{x}_0$

$$= \begin{bmatrix} K & \vdots & \delta u^{off}_0 \end{bmatrix} \begin{bmatrix} \overset{=0}{\delta x_0} \\ 1 \end{bmatrix}$$

$$= \delta u^{off}_0$$



x_0

$\delta x_0 = 0$

$$u_0 = \bar{u}_0 + \delta u_0 = \bar{u}_0 + \delta u_0^{off}$$

$\underline{t=1}$

$$x_1 = f(x_0, u_0) \qquad \delta x_1 = x_1 - \bar{x}_1$$

$$\delta u_1 = \tilde{K}_1 \, \delta x_1 \qquad u_1 = \bar{u}_1 + \delta u_1$$