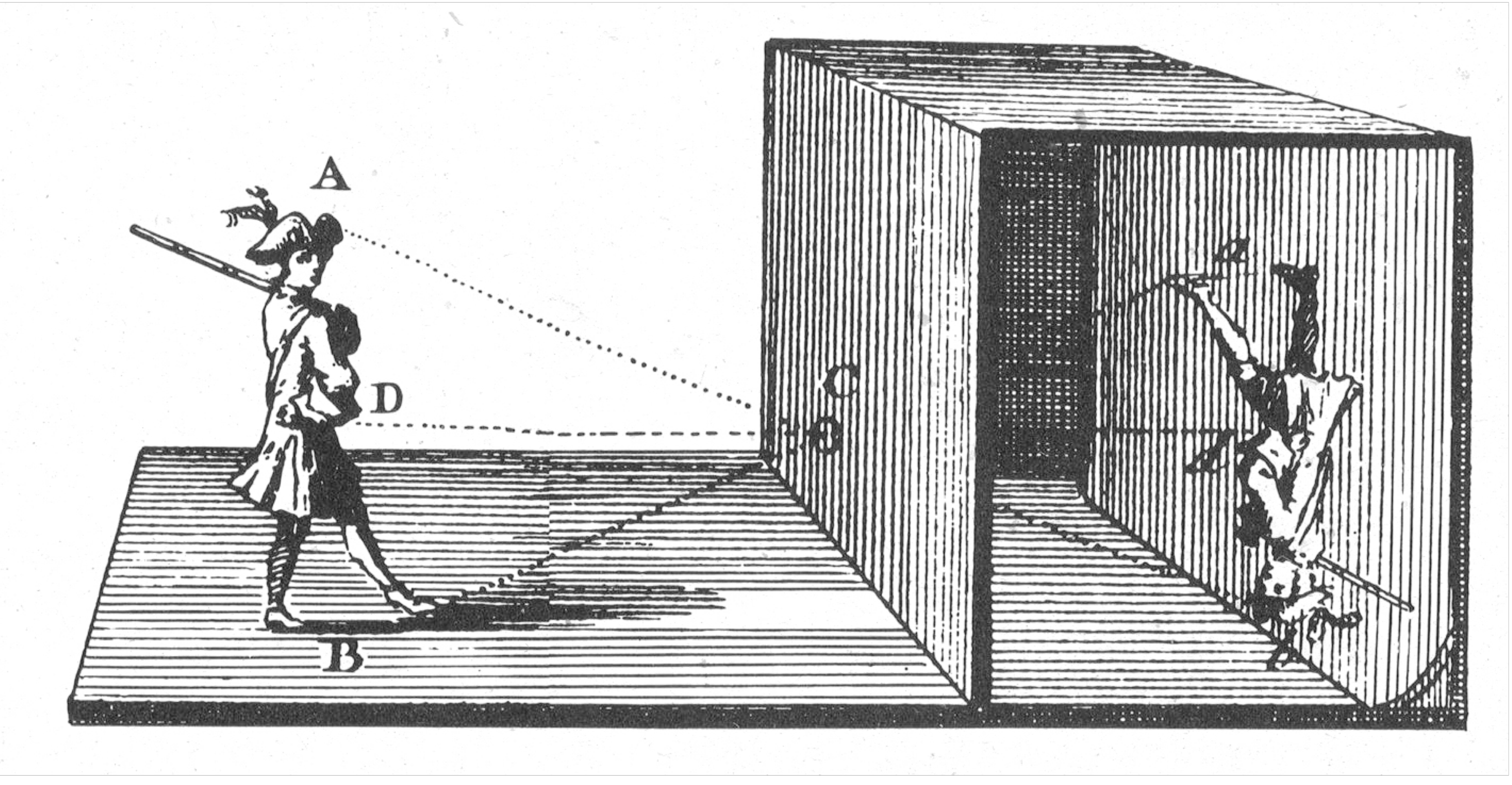
Geometry of Image Formation

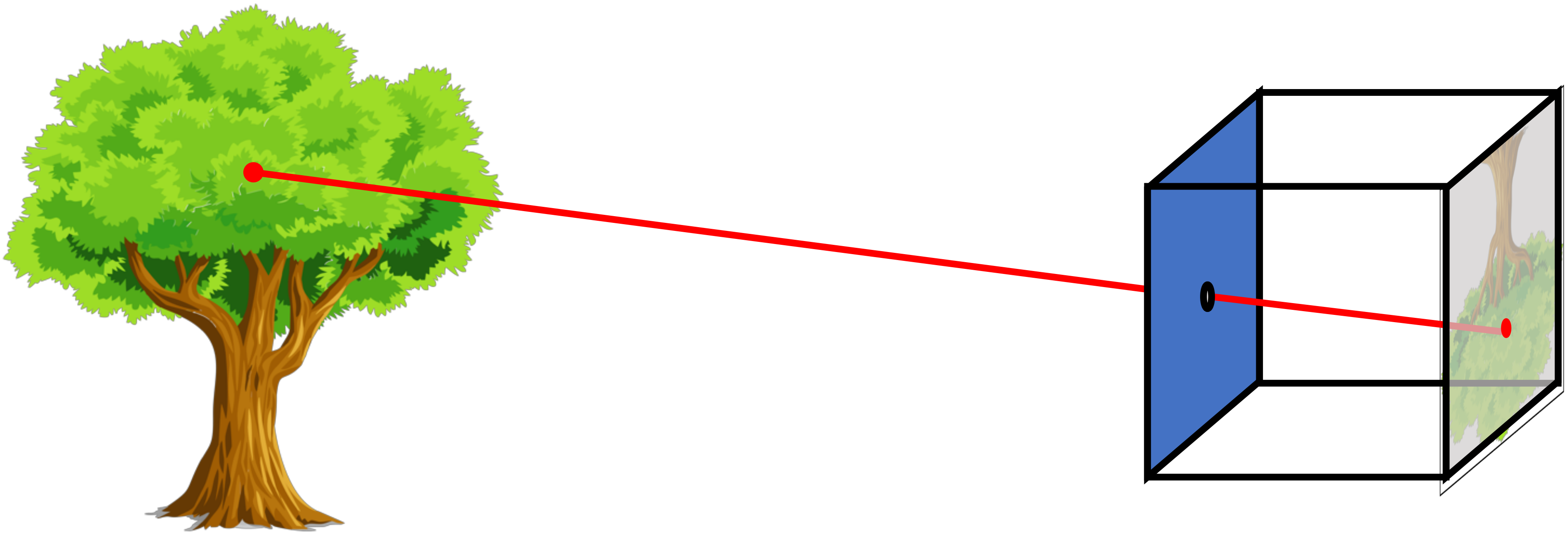
Today

- Geometry of image formation: where a pixel projects in the world
- Deriving perspective effects
- Ways of using perspective effects in recognition

Camera obscura

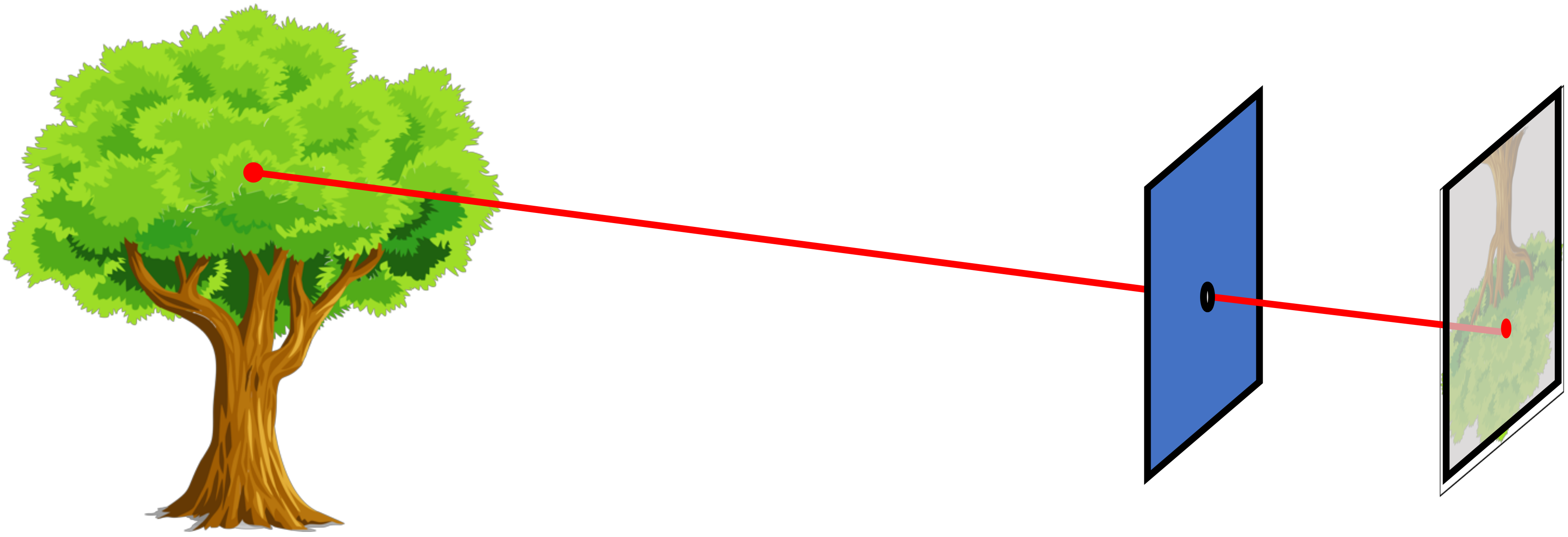


The pinhole camera

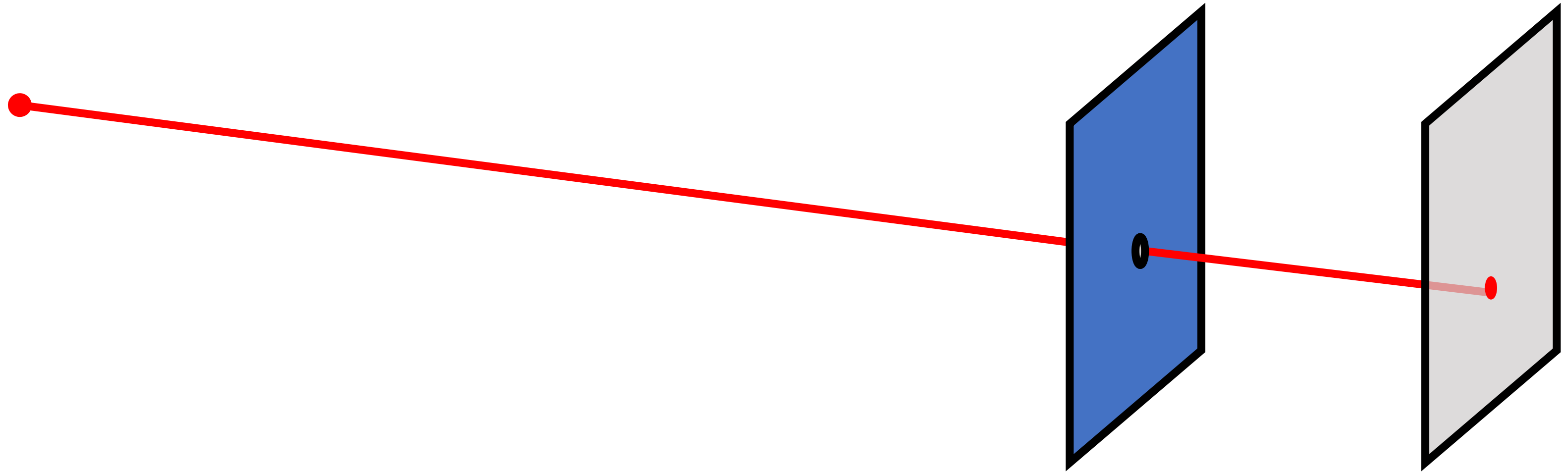


Let's get into the math

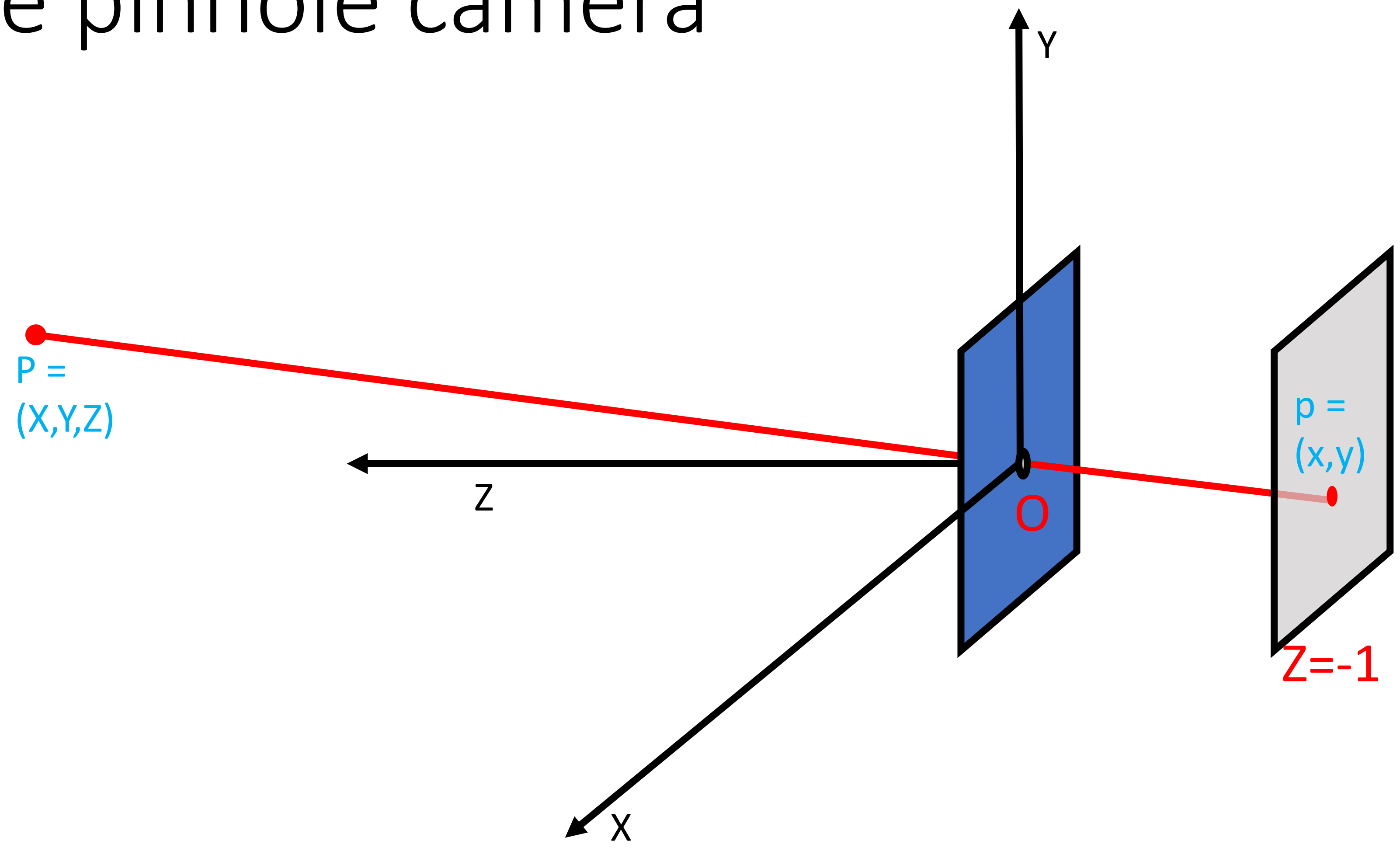
The pinhole camera



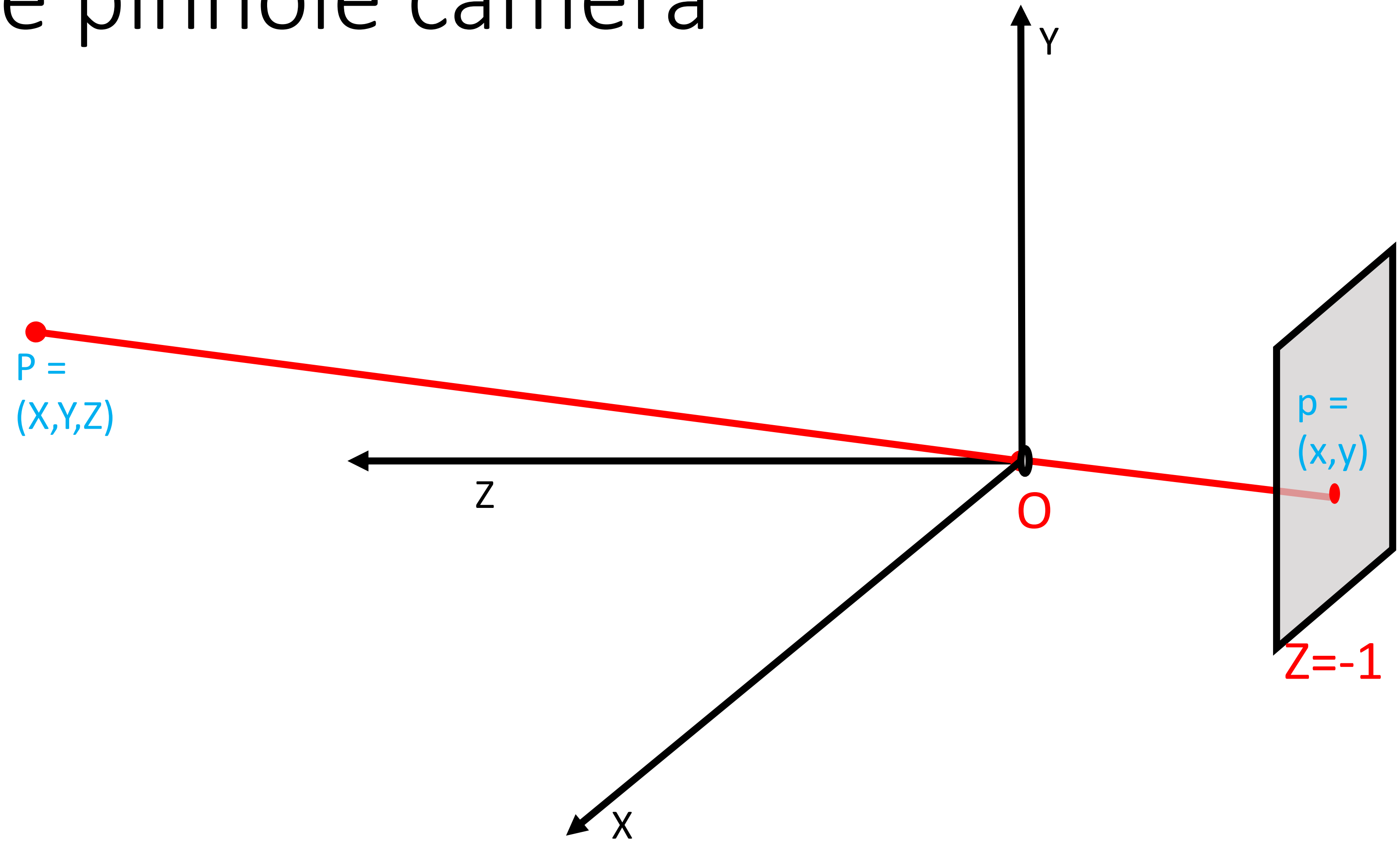
The pinhole camera



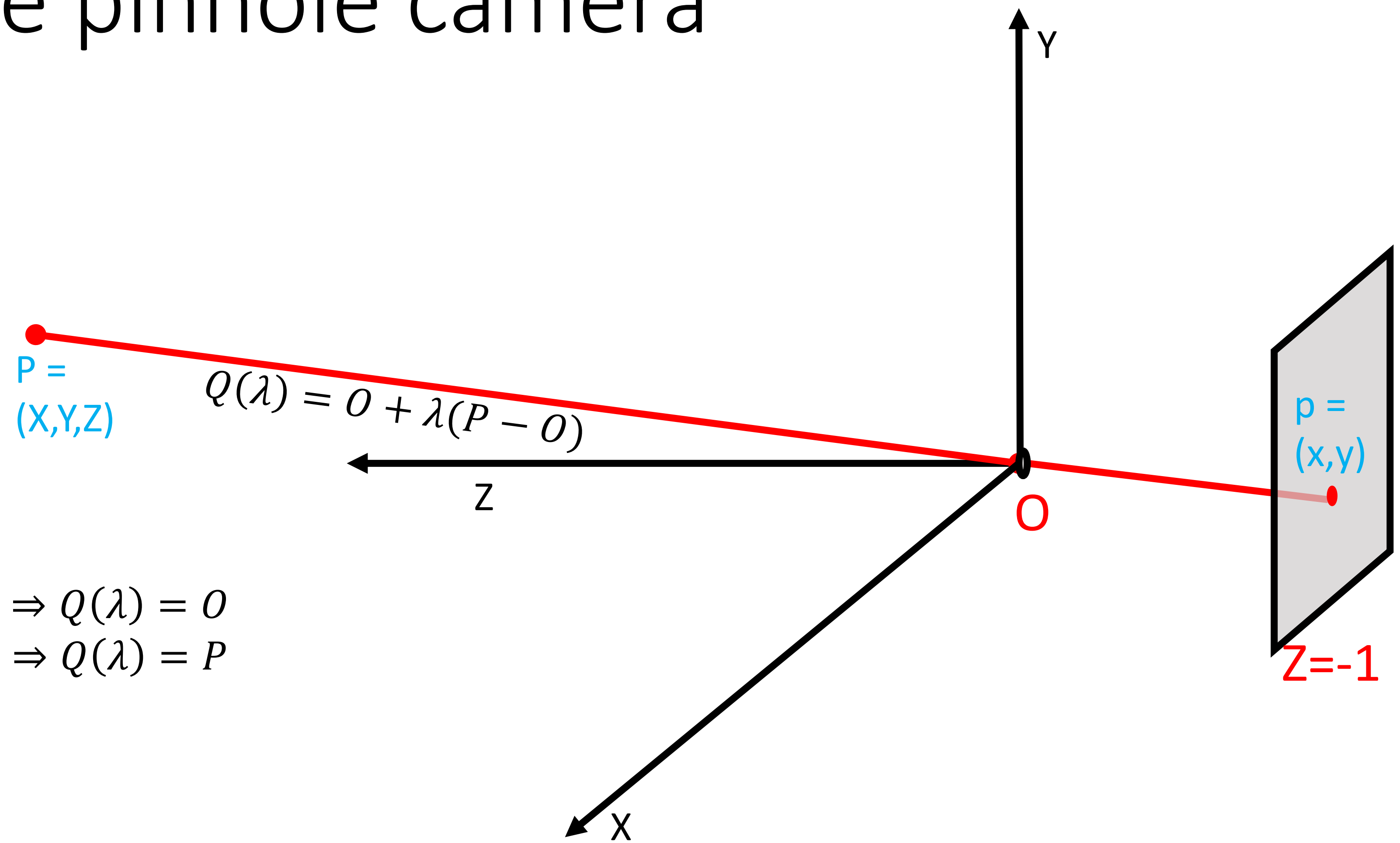
The pinhole camera



The pinhole camera



The pinhole camera



$\lambda = 0 \Rightarrow Q(\lambda) = O$
$\lambda = 1 \Rightarrow Q(\lambda) = P$

$$Q(\lambda) \\ = \big(0 + \lambda(X - 0), 0 + \lambda(Y - 0), 0 + \lambda(Z - 0)\big) \\ = (\lambda X, \lambda Y, \lambda Z)$$

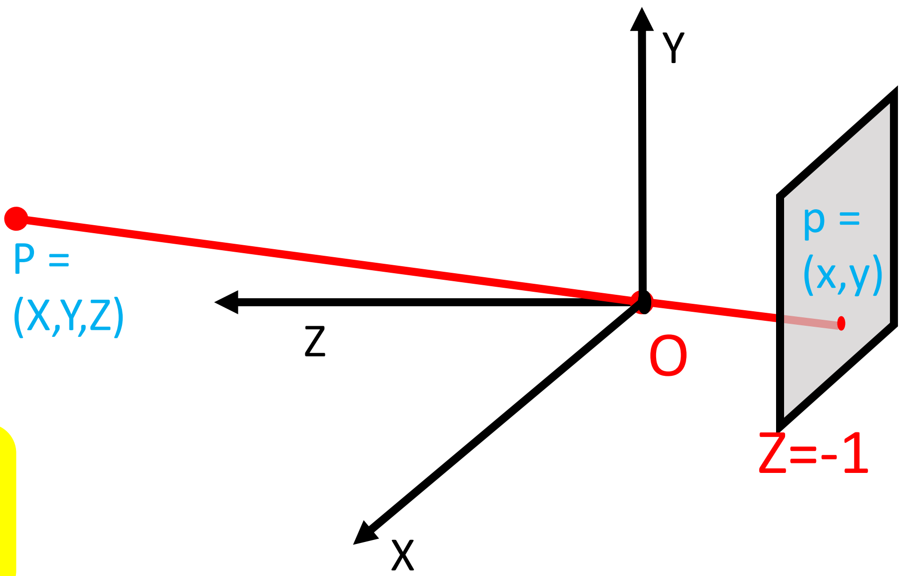
The pinhole camera

- Pinhole camera collapses *ray OP to point p*
- Any point on ray OP = $O + \lambda(P - O) = (\lambda X, \lambda Y, \lambda Z)$
- For this point to lie on Z=-1 plane:

$$\lambda^* Z = -1$$
$$\Rightarrow \lambda^* = \frac{-1}{Z}$$

- Coordinates of point p:

$$(\lambda^* X, \lambda^* Y, \lambda^* Z) = (\frac{-X}{Z}, \frac{-Y}{Z}, -1)$$



The projection equation

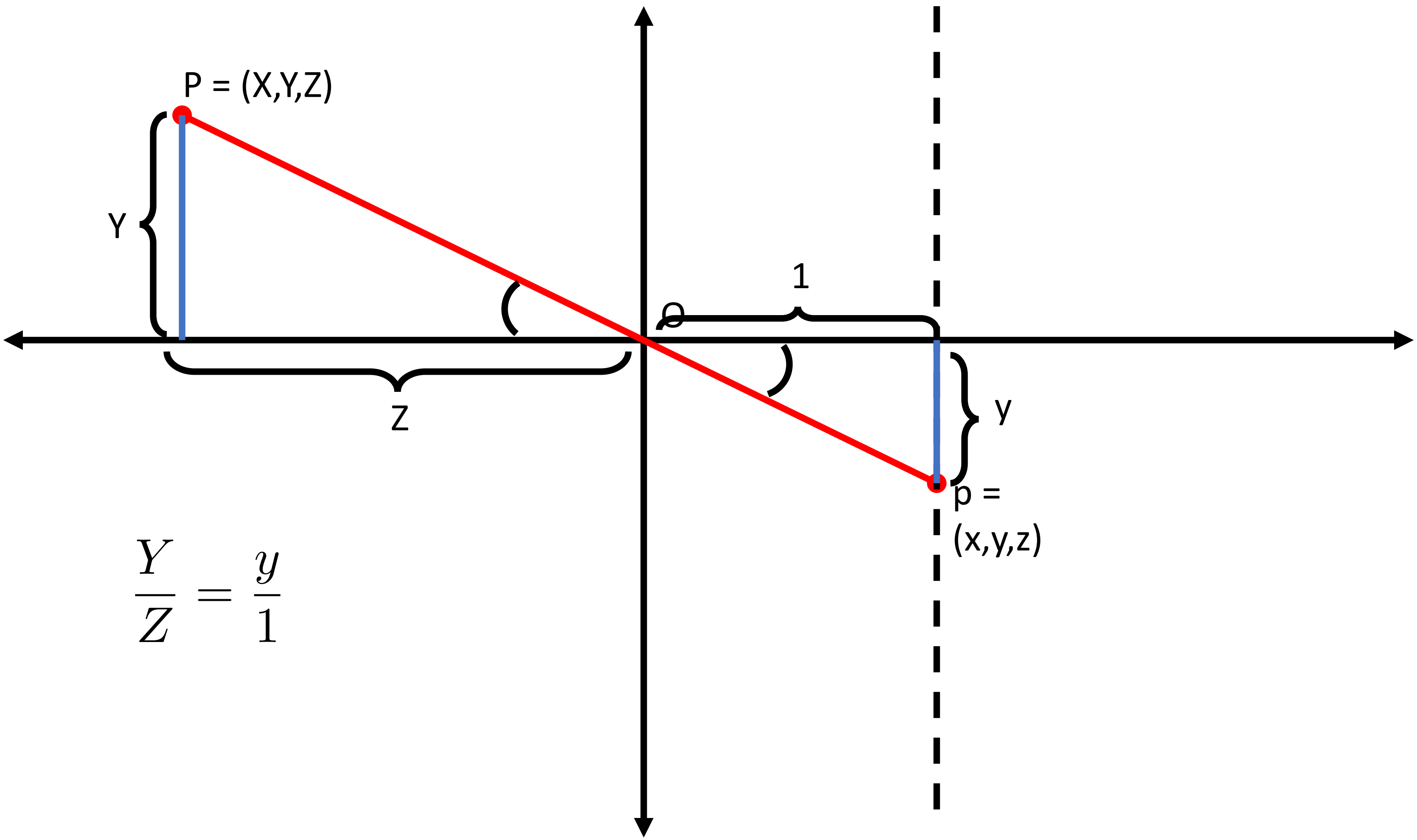
- A point P = (X, Y, Z) in 3D projects to a point p = (x,y) in the image

$$x = \frac{-X}{Z}$$
$$y = \frac{-Y}{Z}$$

- But pinhole camera's image is inverted, invert it back!

$$x = \frac{X}{Z}$$
$$y = \frac{Y}{Z}$$

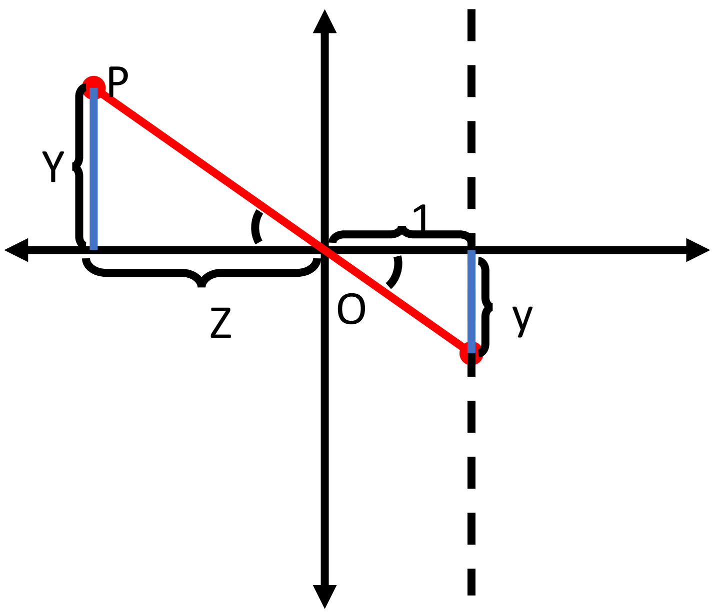
Another derivation

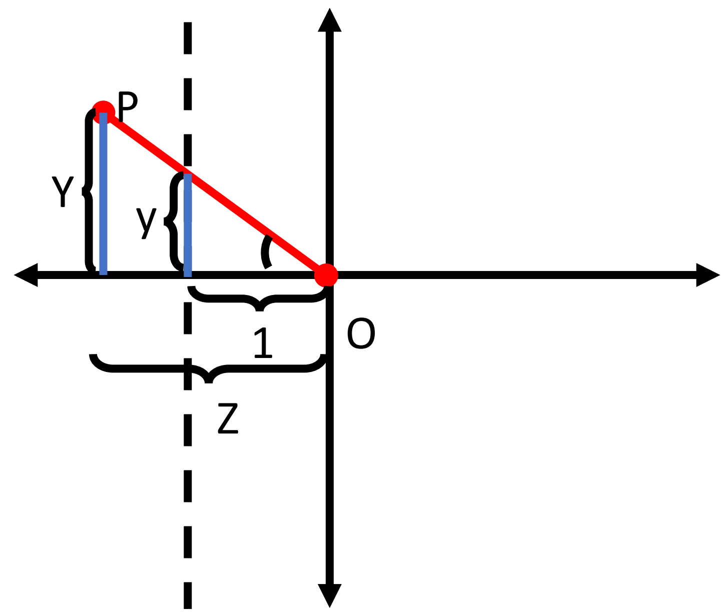


$$\frac{Y}{Z} = \frac{y}{1}$$

A virtual image plane

- A pinhole camera produces an inverted image
- Imagine a ”virtual image plane” in the front of the camera





The projection equation

$$x = \frac{X}{Z}$$

$$y = \frac{Y}{Z}$$

Consequence 1: Farther away objects are smaller

Image of foot: $(\frac{X}{Z}, \frac{Y}{Z})$

Image of head: $(\frac{X}{Z}, \frac{Y+h}{Z})$

$$\frac{Y+h}{Z} - \frac{Y}{Z} = \frac{h}{Z}$$

Consequence 2: Parallel lines converge at a point

- Point on a line passing through point A with direction D:

  $$Q(\lambda) = A + \lambda D$$

- Parallel lines have the same direction but pass through different points

  $$Q(\lambda) = A + \lambda D$$
  $$R(\lambda) = B + \lambda D$$

-

Consequence 2: Parallel lines converge at a point

- Parallel lines have the same direction but pass through different points

$$Q(\lambda) = A + \lambda D$$
$$R(\lambda) = B + \lambda D$$

- $A = (A_X, A_Y, A_Z)$
- $B = (B_X, B_Y, B_Z)$
- $D = (D_X, D_Y, D_Z)$

Consequence 2: Parallel lines converge at a point

- $Q(\lambda) = (A_X + \lambda D_X, A_Y + \lambda D_Y, A_Z + \lambda D_Z)$
- $R(\lambda) = (B_X + \lambda D_X, B_Y + \lambda D_Y, B_Z + \lambda D_Z)$
- $q(\lambda) = \left(\frac{A_X+\lambda D_X}{A_Z+\lambda D_Z}, \frac{A_Y+\lambda D_Y}{A_Z+\lambda D_Z}\right)$
- $r(\lambda) = \left(\frac{B_X+\lambda D_X}{B_Z+\lambda D_Z}, \frac{B_Y+\lambda D_Y}{B_Z+\lambda D_Z}\right)$
- Need to look at these points as Z goes to infinity
- Same as $\lambda \to \infty$

Consequence 2: Parallel lines converge at a point

- $q(\lambda) = \left(\frac{A_X+\lambda D_X}{A_Z+\lambda D_Z}, \frac{A_Y+\lambda D_Y}{A_Z+\lambda D_Z}\right)$
- $r(\lambda) = \left(\frac{B_X+\lambda D_X}{B_Z+\lambda D_Z}, \frac{B_Y+\lambda D_Y}{B_Z+\lambda D_Z}\right)$

$$\lim_{\lambda\to\infty} \frac{A_X + \lambda D_X}{A_Z + \lambda D_Z} = \lim_{\lambda\to\infty} \frac{\frac{A_X}{\lambda} + D_X}{\frac{A_Z}{\lambda} + D_Z} = \frac{D_X}{D_Z}$$

$$\lim_{\lambda\to\infty} q(\lambda) = (\frac{D_X}{D_Z}, \frac{D_Y}{D_Z})$$

$$\lim_{\lambda\to\infty} r(\lambda) = (\frac{D_X}{D_Z}, \frac{D_Y}{D_Z})$$

Consequence 2: Parallel lines converge at a point

- Parallel lines have the same direction but pass through different points

$$Q(\lambda) = A + \lambda D$$
$$R(\lambda) = B + \lambda D$$

- Parallel lines converge at the same point $(\frac{D_X}{D_Z}, \frac{D_Y}{D_Z})$
- This point of convergence is called the *vanishing point*
- What happens if $D_Z = 0$?

Consequence 2: Parallel lines converge at a point

What about planes?

$$N_X X + N_Y Y + N_Z Z = d$$

$$\Rightarrow N_X \frac{X}{Z} + N_Y \frac{Y}{Z} + N_Z = \frac{d}{Z}$$

$$\Rightarrow N_X x + N_Y y + N_Z = \frac{d}{Z}$$

Take the limit as Z approaches infinity

$$N_X x + N_Y y + N_Z = 0$$

Vanishing line of a plane

What about planes?

$$N_X X + N_Y Y + N_Z Z = d$$

Normal: (N_X, N_Y, N_Z)

What do parallel planes look like?

$$N_X X + N_Y Y + N_Z Z = d$$

$$N_X x + N_Y y + N_Z = 0$$

$$N_X X + N_Y Y + N_Z Z = c$$

$$N_X x + N_Y y + N_Z = 0$$

Vanishing lines

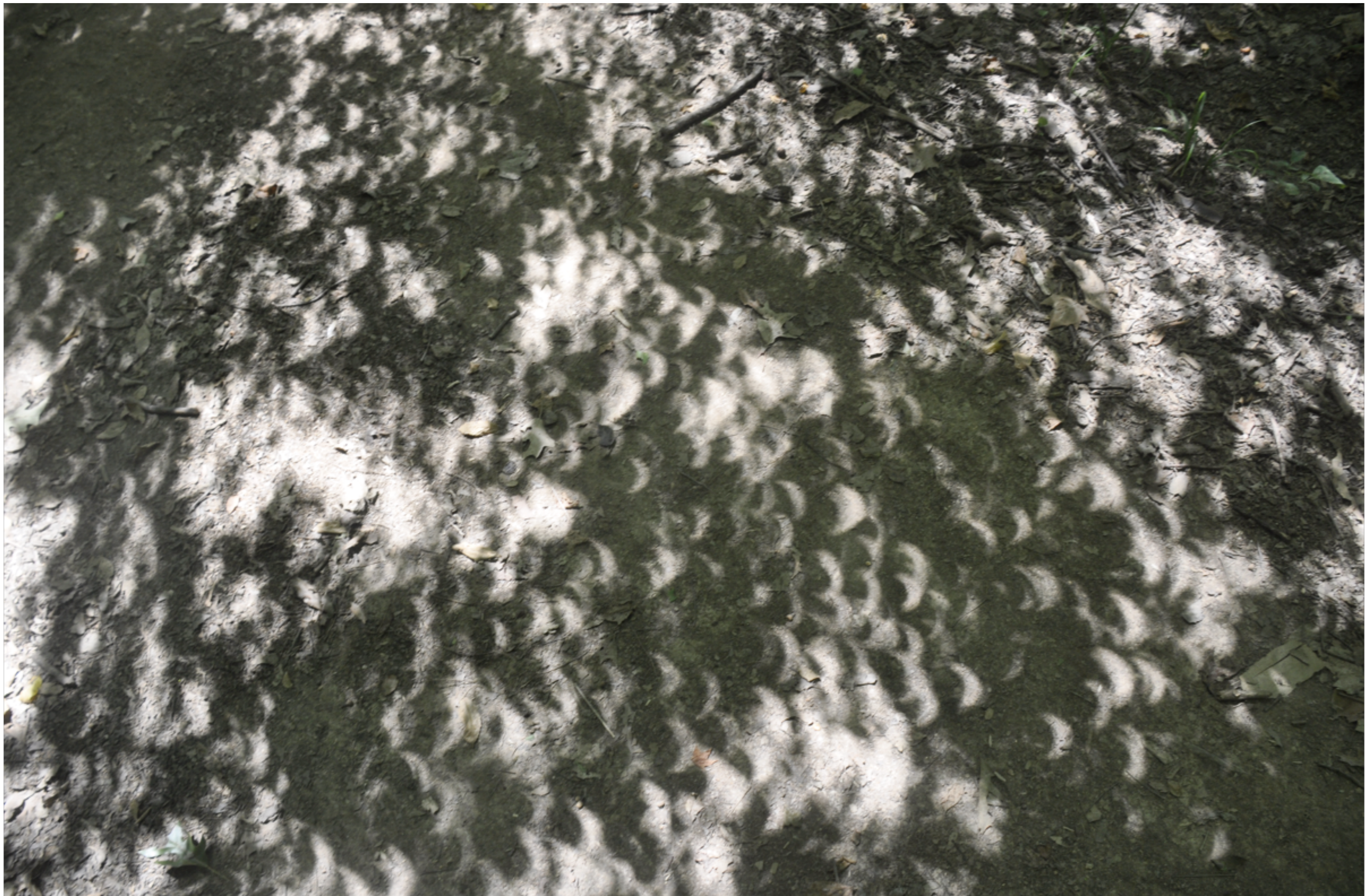
Parallel planes converge!

Vanishing line

$$N_X X + N_Y Y + N_Z Z = d$$

- What happens if $N_X = N_Y = 0$?
- Equation of the plane: $Z = c$
- Vanishing line?

Accidental pinholes



Accidental pinholes

Torralba, Antonio, and William T. Freeman. "Accidental pinhole and pinspeck cameras: Revealing the scene outside the picture." *Computer Vision and Pattern Recognition (CVPR), 2012 IEEE Conference on*. IEEE, 2012.

Accidental pinholes

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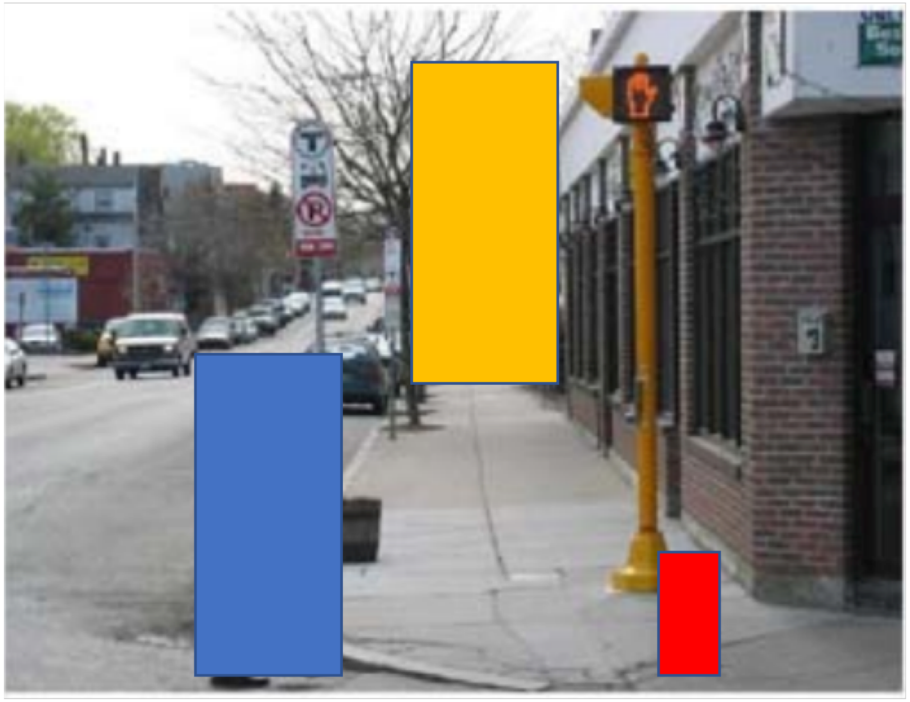
Torralba, Antonio, and William T. Freeman. "Accidental pinhole and pinspeck cameras: Revealing the scene outside the picture." *Computer Vision and Pattern Recognition (CVPR), 2012 IEEE Conference on*. IEEE, 2012.

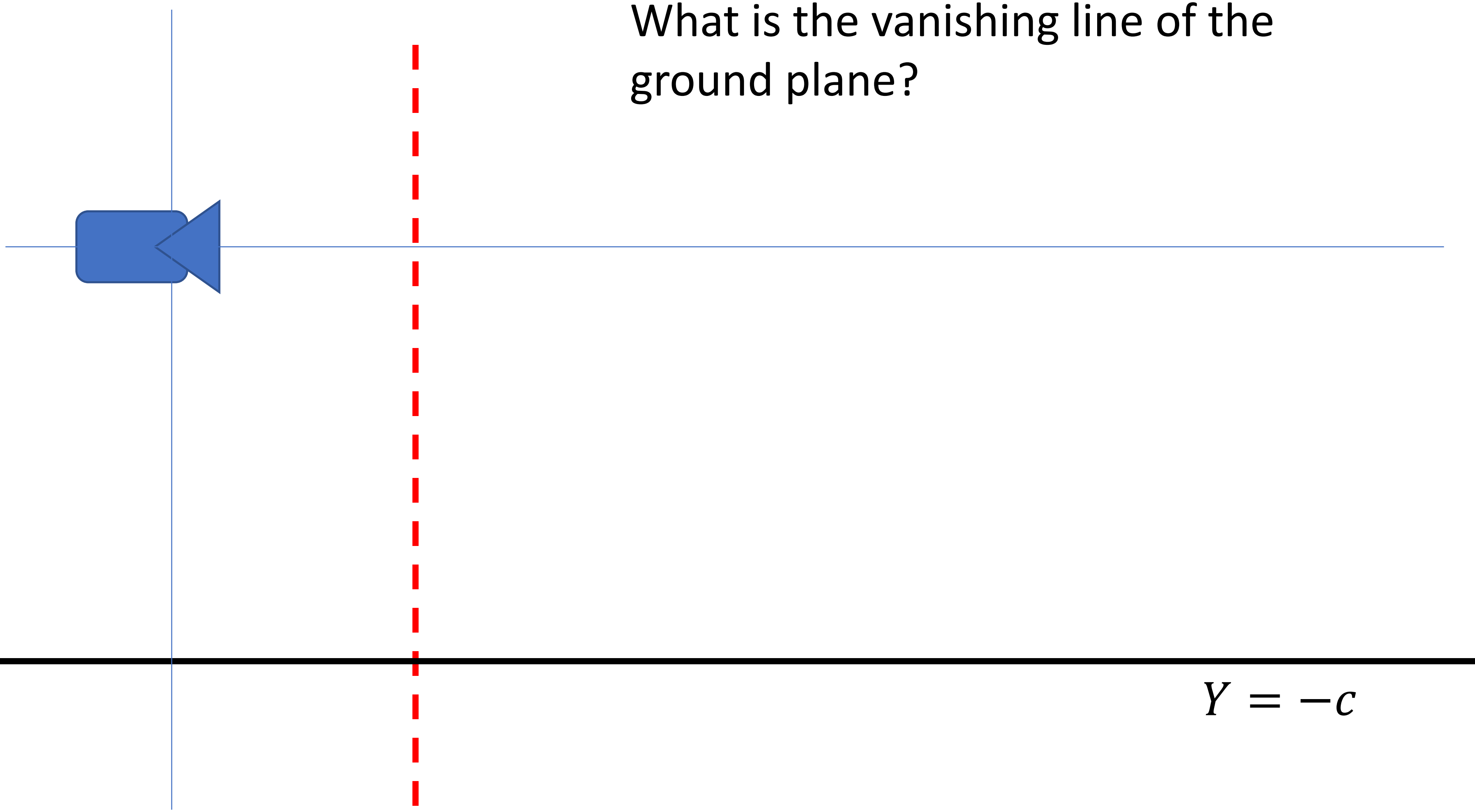
Geometry for recognition

- Which of these is likely to be an adult human?

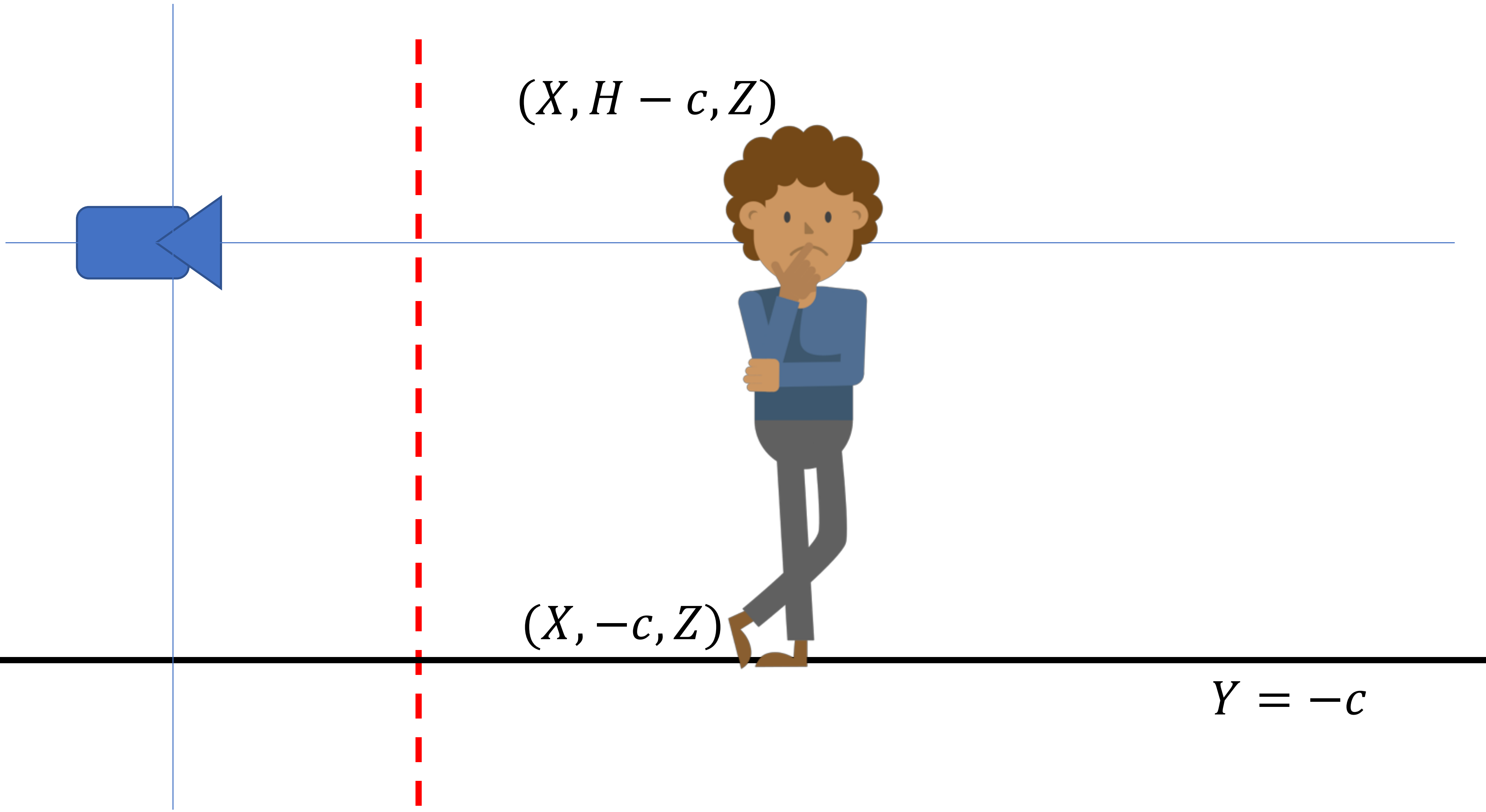


The math

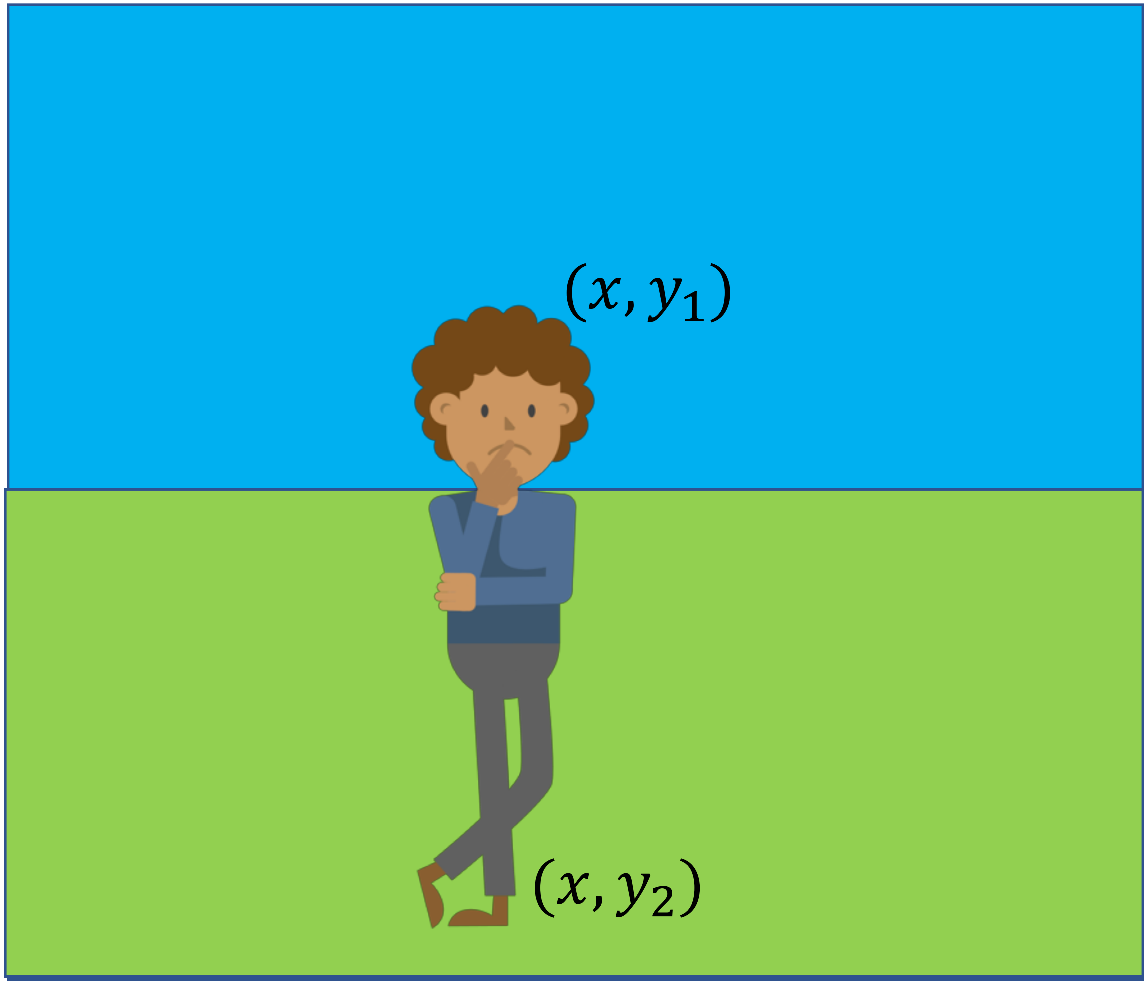
What is the vanishing line of the ground plane?



The math



The math



The math

$$y_2 = \frac{-c}{Z}$$

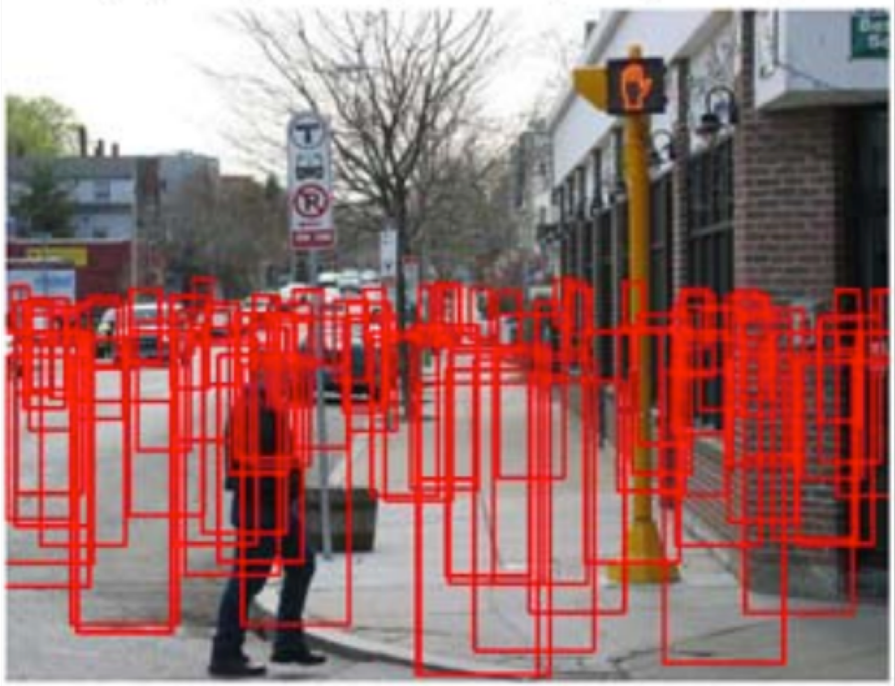
$$\Longrightarrow Z = \frac{-c}{y_2}$$

$$y_1 = \frac{H - c}{Z}$$

$$\Longrightarrow H = Z\, y_1 + c$$

$$\Longrightarrow H = \frac{c(y_2 - y_1)}{y_2} = \frac{ch}{|y_2|}$$

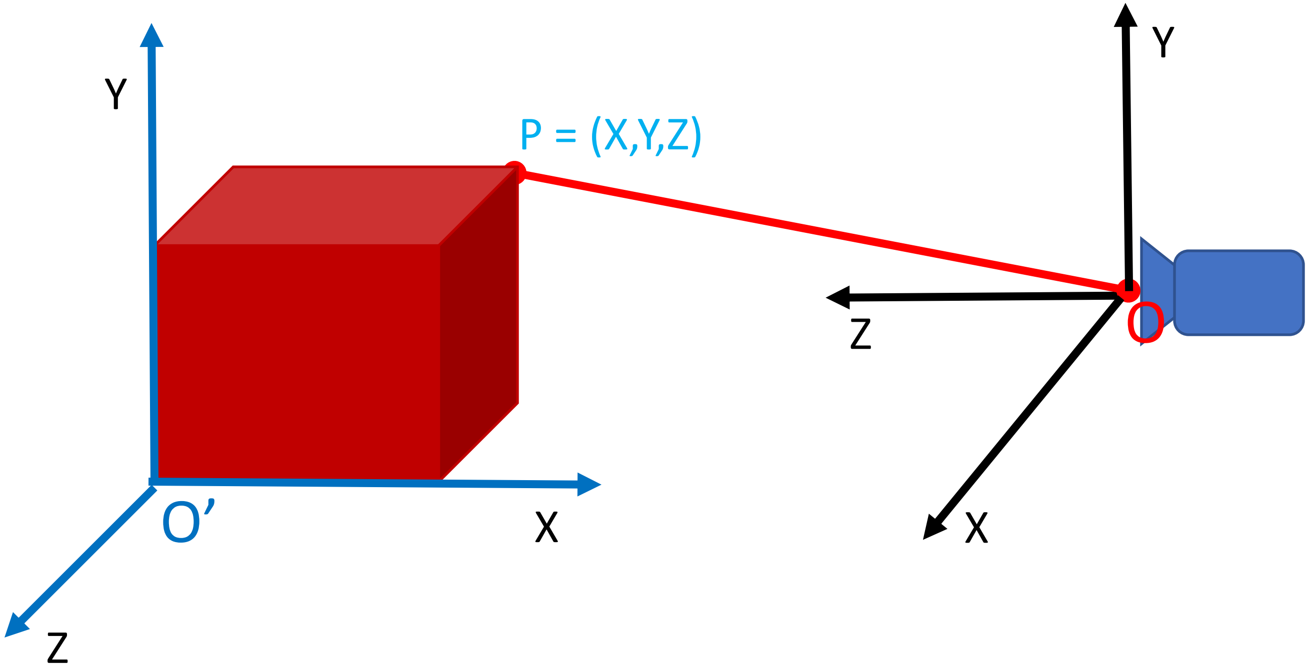
Geometry for recognition



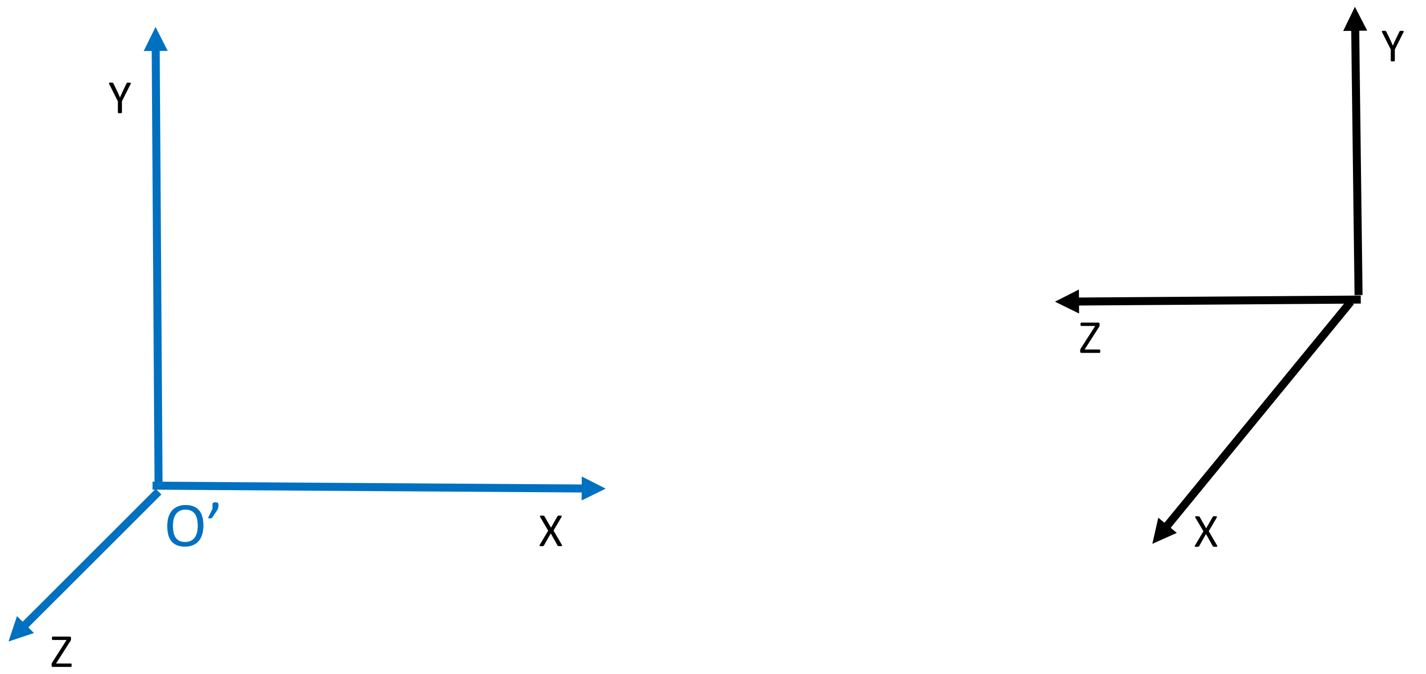
(f) P(person | viewpoint)

Hoiem, Derek, Alexei A. Efros, and Martial Hebert. "Putting objects in perspective." *International Journal of Computer Vision* 80.1 (2008): 3-15.

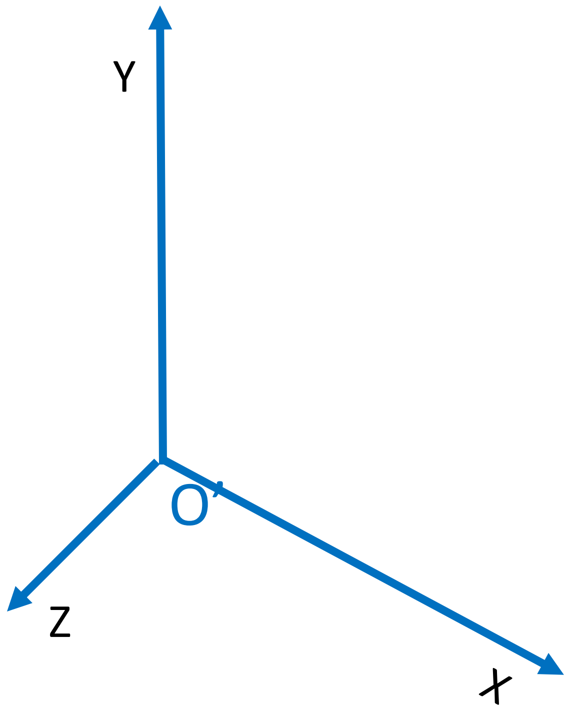
Changing coordinate systems

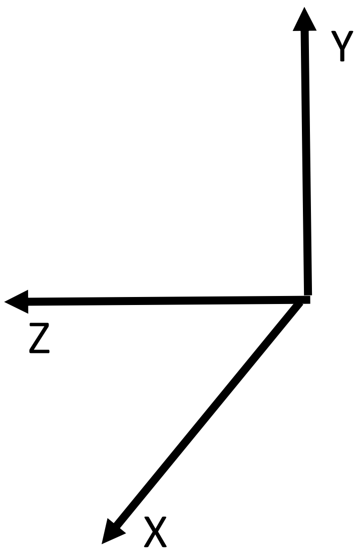


Changing coordinate systems

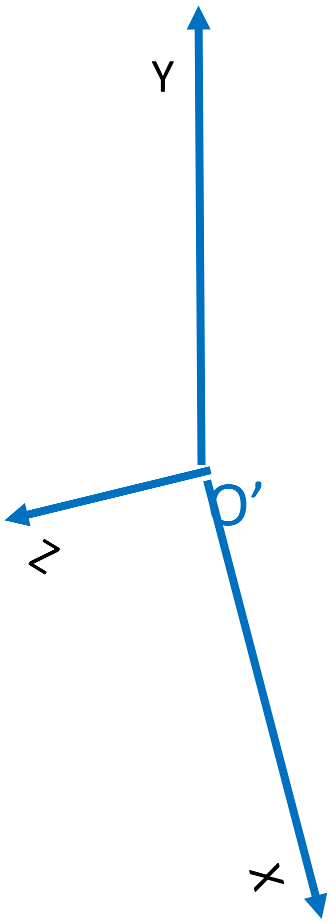


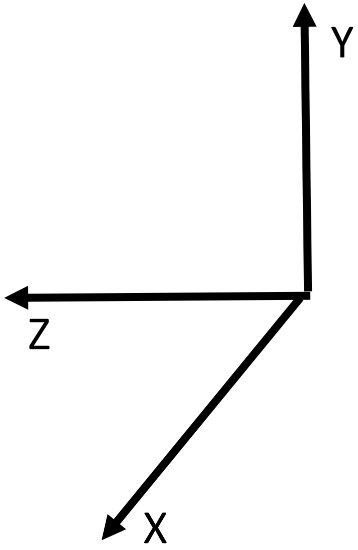
Changing coordinate systems



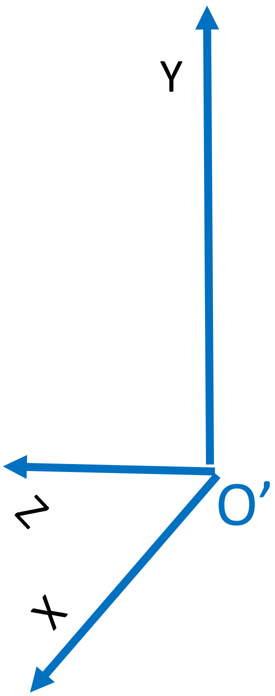


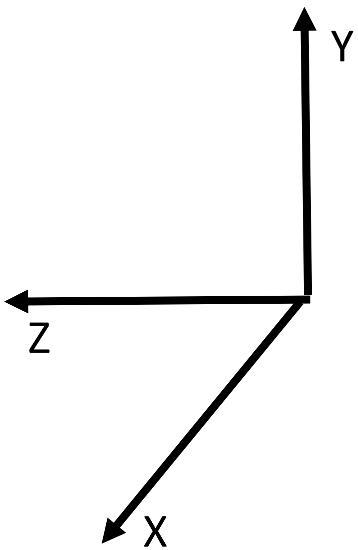
Changing coordinate systems



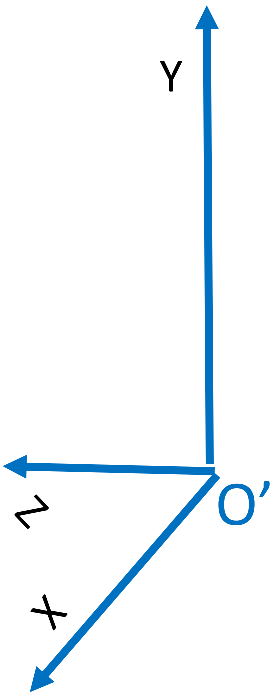


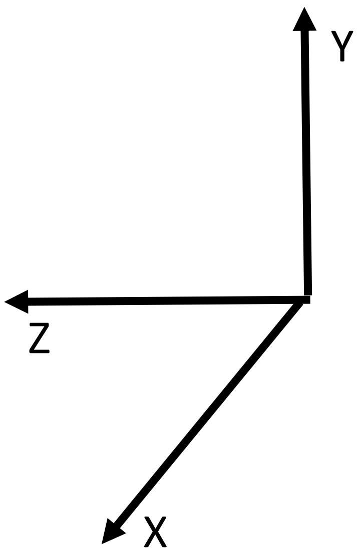
Changing coordinate systems





Changing coordinate systems





Rotations and translations

- How do you represent a rotation?
- A point in 3D: (X,Y,Z)
- Rotations can be represented as a matrix multiplication

$$\mathbf{v}' = R\mathbf{v}$$

- What are the properties of rotation matrices?

Properties of rotation matrices

- Rotation does not change the length of vectors

$$\mathbf{v}' = R\mathbf{v}$$

$$\|\mathbf{v}'\|^2 = \mathbf{v}'^T\mathbf{v}'$$

$$= \mathbf{v}^T R^T R\mathbf{v}$$

$$\|\mathbf{v}\|^2 = \mathbf{v}^T\mathbf{v}$$

$$\Rightarrow R^T R = I$$

Properties of rotation matrices

$$\Rightarrow R^T R = I$$

$$\Rightarrow det(R)^2 = 1$$

$$\Rightarrow det(R) = \pm 1$$

$$det(R) = 1$$

Rotation

$$det(R) = -1$$

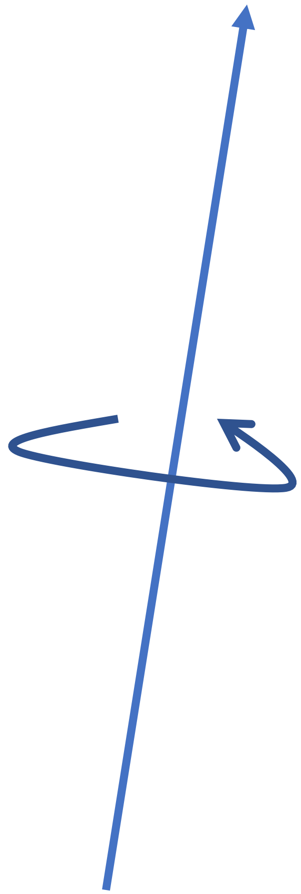
Reflection

Rotation matrices

- Rotations in 3D have an axis and an angle
- Axis: vector that does not change when rotated

$$R\mathbf{v} = \mathbf{v}$$

- Rotation matrix has eigenvector that has eigenvalue 1



Rotation matrices from axis and angle

- Rotation matrix for rotation about axis $\boldsymbol{v}$ and θ
- First define the following matrix

$$[\mathbf{v}]_\times = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}$$

- Interesting fact: this matrix represents cross product

$$[\mathbf{v}]_\times \mathbf{x} = \mathbf{v} \times \mathbf{x}$$

Rotation matrices from axis and angle

- Rotation matrix for rotation about axis $\boldsymbol{v}$ and θ
- Rodrigues' formula for rotation matrices

$$R = I + (\sin\theta)[\mathbf{v}]_\times + (1 - \cos\theta)[\mathbf{v}]_\times^2$$

Translations

$$\mathbf{x}' = \mathbf{x} + \mathbf{t}$$

- Can this be written as a matrix multiplication?

Putting everything together

- Change coordinate system so that center of the coordinate system is at pinhole and Z axis is along viewing direction

$$\mathbf{x}'_w = R\mathbf{x}_w + \mathbf{t}$$

- Perspective projection

$$\mathbf{x}'_w \equiv (X, Y, Z)$$

$$\mathbf{x}'_{img} \equiv (x, y)$$

$$x = \frac{X}{Z}$$

$$y = \frac{Y}{Z}$$

The projection equation

$$x = \frac{X}{Z}$$
$$y = \frac{Y}{Z}$$

- Is this equation linear?
- Can this equation be represented by a matrix multiplication?

Is projection linear?

$$X' = aX + b$$
$$Y' = aY + b$$
$$Z' = aZ + b$$

$$x' = \frac{aX + b}{aZ + b}$$

$$y' = \frac{aY + b}{aZ + b}$$

Can projection be represented as a matrix multiplication?

Matrix multiplication

$$\begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} aX + bY + cZ \\ pX + qY + rZ \end{bmatrix}$$

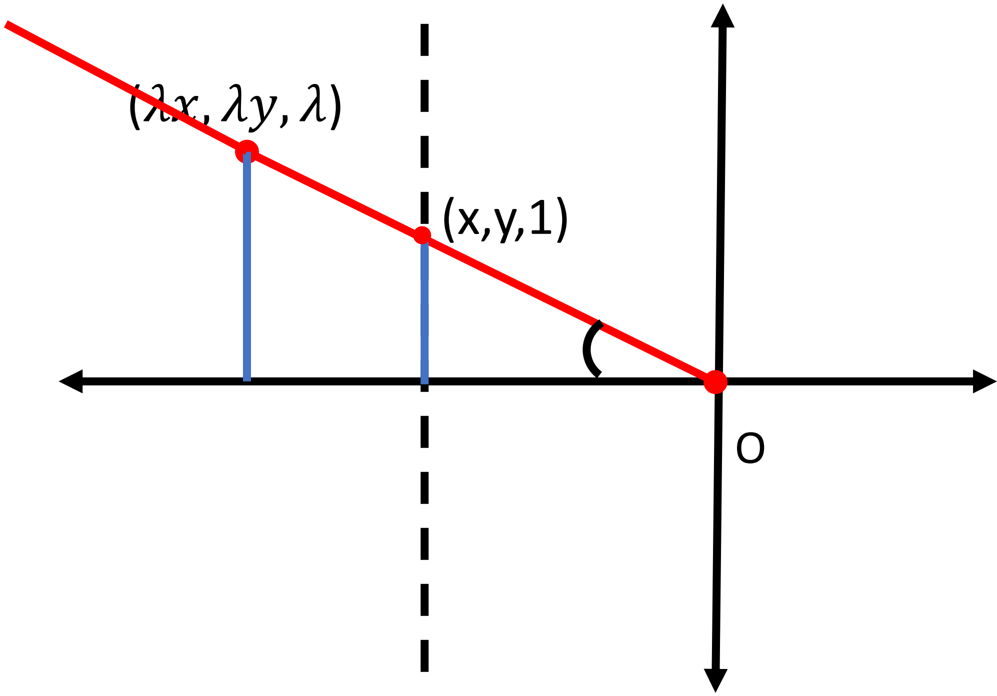
Perspective projection

$$x = \frac{X}{Z}$$

$$y = \frac{Y}{Z}$$

The space of rays

- Every point on a ray maps it to a point on image plane
- Perspective projection maps rays to points
- All points $(\lambda x, \lambda y, \lambda)$ map to the same image point (x,y,1)



Projective space

- Standard 2D space (plane) $\mathbb{R}^2$: Each point represented by 2 coordinates (x,y)
- Projective 2D space (plane) $\mathbb{P}^2$: Each "point" represented by *3* coordinates (x,y,z), BUT:
  - $(\lambda x, \lambda y, \lambda z) \equiv (x, y, z)$

- Mapping $\mathbb{R}^2$ to $\mathbb{P}^2$ (points to rays):

$$(x, y) \rightarrow (x, y, 1)$$

- Mapping $\mathbb{P}^2$ to $\mathbb{R}^2$ (rays to points):

$$(x, y, z) \rightarrow (\frac{x}{z}, \frac{y}{z})$$

Projective space and homogenous coordinates

- Mapping $\mathbb{R}^2$ to $\mathbb{P}^2$ (points to rays):

$$(x, y) \rightarrow (x, y, 1)$$

- Mapping $\mathbb{P}^2$ to $\mathbb{R}^2$ (rays to points):

$$(x, y, z) \rightarrow (\frac{x}{z}, \frac{y}{z})$$

- A change of coordinates
- Also called *homogenous coordinates*

Homogenous coordinates

- In standard Euclidean coordinates
  - 2D points : (x,y)
  - 3D points : (x,y,z)

- In homogenous coordinates
  - 2D points : (x,y,1)
  - 3D points : (x,y,z,1)

Why homogenous coordinates?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \equiv \begin{bmatrix} \frac{X}{Z} \\ \frac{Y}{Z} \\ 1 \end{bmatrix}$$

Homogenous coordinates of world point

Homogenous coordinates of image point

Why homogenous coordinates?

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}X\\Y\\Z\\1\end{bmatrix} = \begin{bmatrix}X\\Y\\Z\end{bmatrix} \equiv \begin{bmatrix}\frac{X}{Z}\\\frac{Y}{Z}\\1\end{bmatrix}$$

$$P\vec{\mathbf{x}}_w = \vec{\mathbf{x}}_{img}$$

- Perspective projection is matrix multiplication in homogenous coordinates!

Why homogenous coordinates?

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Translation is matrix multiplication in homogenous coordinates!

Homogenous coordinates

$$\begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} aX + bY + cZ + t_x \\ dX + eY + fZ + t_y \\ gX + hY + iZ + t_z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} = \begin{bmatrix} M\mathbf{x}_w + \mathbf{t} \\ 1 \end{bmatrix}$$

Homogenous coordinates

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \equiv \begin{bmatrix} \frac{X}{Z} \\ \frac{Y}{Z} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} I & \mathbf{0} \end{bmatrix}$$

Perspective projection in homogenous coordinates

$$\vec{\mathbf{x}}_{img} = \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img} = \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

More about matrix transformations

$$\begin{bmatrix} I & \mathbf{0} \end{bmatrix}$$ 3 x 4 : Perspective projection

$$\begin{bmatrix} I & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$ 4 x 4 : Translation

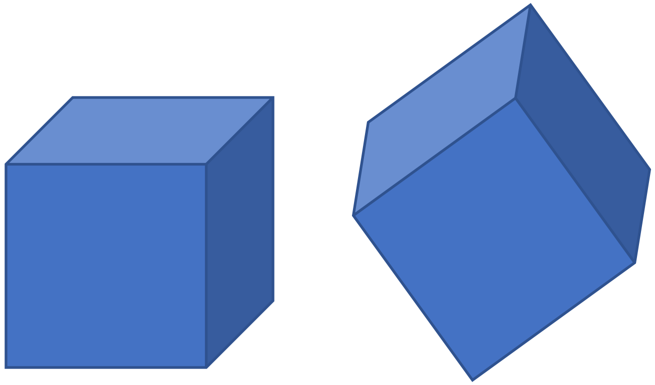
$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$ 4 x 4 : Affine transformation (linear transformation + translation)

More about matrix transformations

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$M^T M = I$$

Euclidean



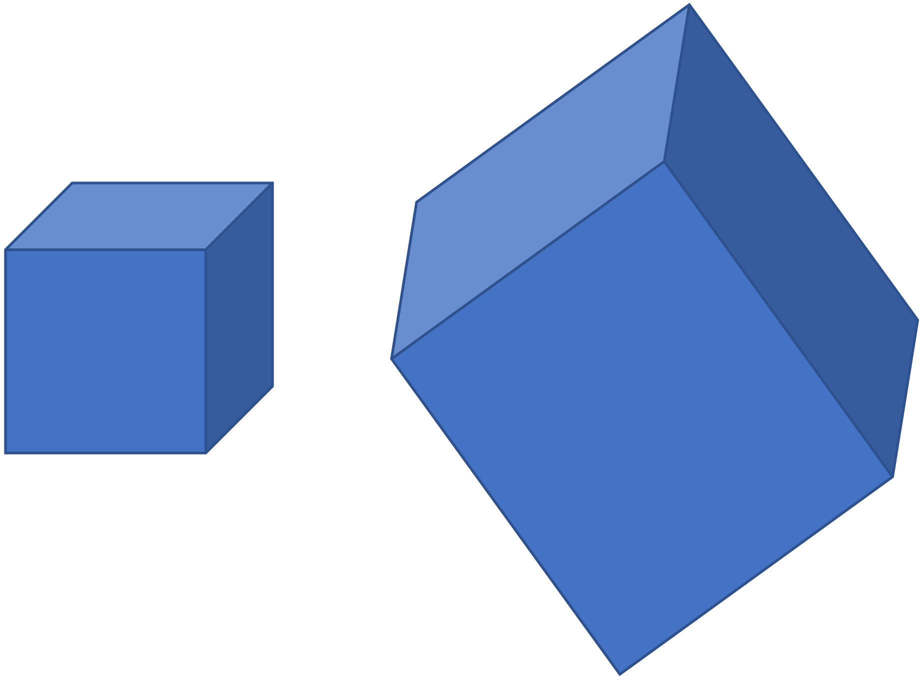
More about matrix transformations

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$M = sR$$

$$R^T R = I$$

Similarity transformation

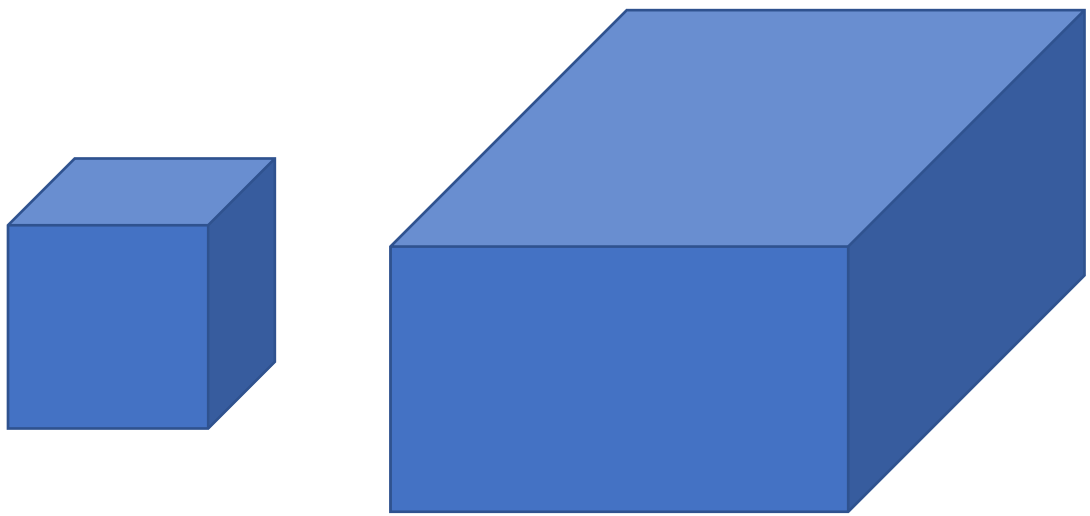


More about matrix transformations

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

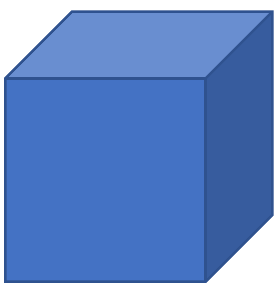
Anisotropic scaling and translation

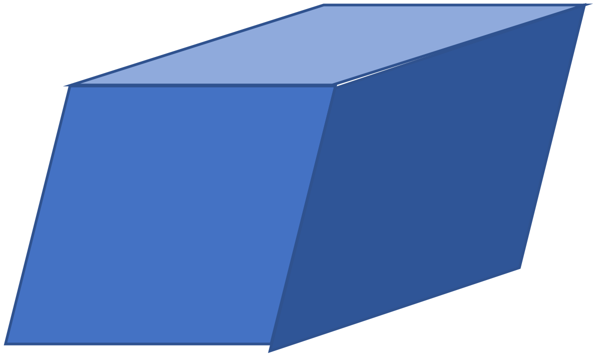


More about matrix transformations

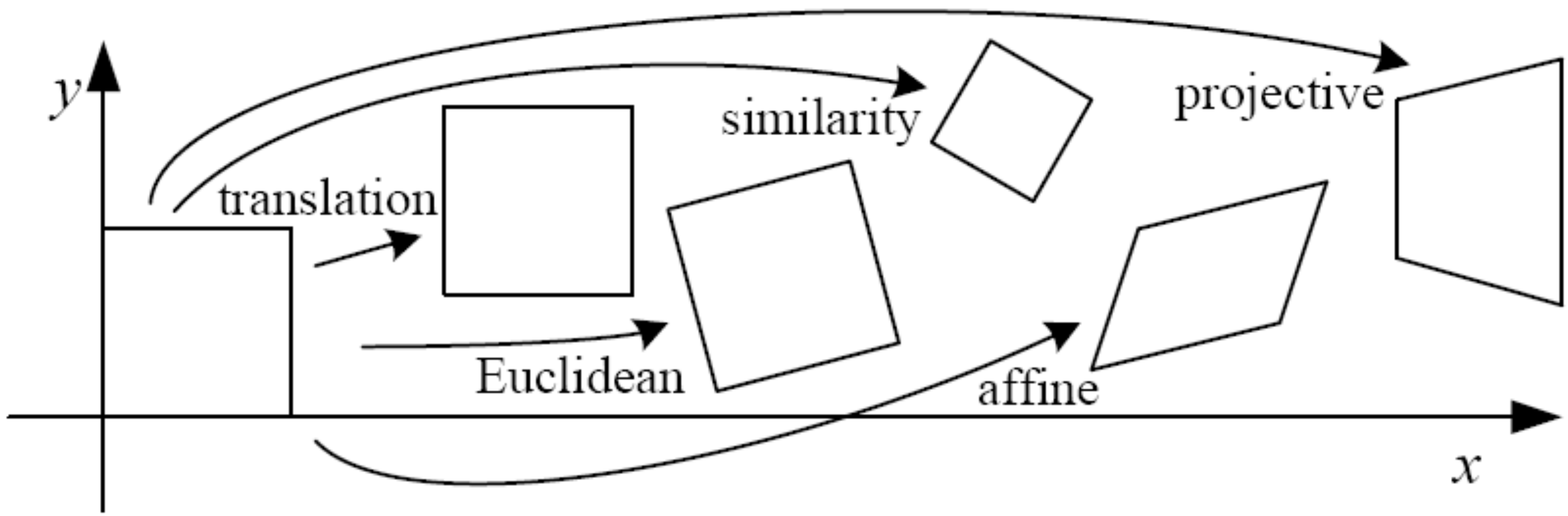
$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

General affine transformation





Matrix transformations in 2D



Perspective projection in homogenous coordinates

$$\vec{\mathbf{x}}_{img} \equiv \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img} \equiv \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Matrix transformations in 2D

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

$$K = \begin{bmatrix} 1 & 0 & t_u \\ 0 & 1 & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

$$K = \begin{bmatrix} s_x & 0 & t_u \\ 0 & s_y & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling of Image x and y (conversion from "meters" to "pixels")

$$K = \begin{bmatrix} s_x & \alpha & t_u \\ 0 & s_y & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

Added skew if image x and y axes are not perpendicular

Final perspective projection

Camera extrinsics: where your camera is relative to the world. Changes if you move the camera

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Camera intrinsics: how your camera handles pixel. Changes if you change your camera

$$\vec{\mathbf{x}}_{img} \equiv P \vec{\mathbf{x}}_w$$

Final perspective projection

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Camera parameters

$$\vec{\mathbf{x}}_{img} \equiv P \vec{\mathbf{x}}_w$$