All about rotations

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Let's delve deeper into the first coordinate transformation, a translation and a rotation, since this is something that will come up frequently.

Rotations and translations belong to a class of transformations called "rigid transformations". Rigid transformations do not change the distance between two points. Thus they do not squeeze or stretch any objects in any way.

It is fairly clear that translations will not change the distance between two points. But what about rotations?

In fact, rotations are *defined* as linear transformations that do not change the distance between points.

$$
\begin{aligned}
\|\mathbf{R}x - \mathbf{R}y\|^2 &= \|x - y\|^2 \forall x, y \\
\Rightarrow (x-y)^T \mathbf{R}^T \mathbf{R}(x-y) &= (x-y)^T (x-y) \forall x, y \\
&\Rightarrow \mathbf{R}^T \mathbf{R} = \mathbf{I}
\end{aligned} \tag{1}
$$

Thus, any matrix R for which $R^T R = I$ will maintain distance between points. Such matrices are called orthonormal matrices.

$\mathbf{R}^T \mathbf{R} = I \rightarrow \det(\mathbf{R})^2 = 1$. Thus, orthonormal matrices have a determinant of 1 or -1. Rotations are a special subset of orthonormal matrices in that they have a determinant of 1. Transformations with a negative determinant change the *handedness* of the coordinate system. Thus rotations are *linear transformations that preserve both distances and handedness.*

Rotations have several properties:

1. They form a group. The identity transformation is a trivial rotation. Composing two rotations by multiplying together also leads to a rotation. To see this note that:

   (a) $\mathbf{R}_2^T \mathbf{R}_1^T \mathbf{R}_1 \mathbf{R}_2 = \mathbf{R}_2^T \mathbf{R}_2 = \mathbf{I}$, and

   (b) $\det(\mathbf{R}_1 \mathbf{R}_2) = \det(\mathbf{R}_1)\det(\mathbf{R}_2) = 1$.

   In 3D space, this group is called the *special orthogonal group* of order 3, $SO(3)$, special because they preserve handedness.

2. A theorem by Euler states that every 3D rotation corresponds to rotation by an angle about an axis. The axis of a rotation is the line which is unchanged by the rotation, i.e, points $\lambda\mathbf{x}$ s.t. $\mathbf{R}\mathbf{x} = \mathbf{x}$.

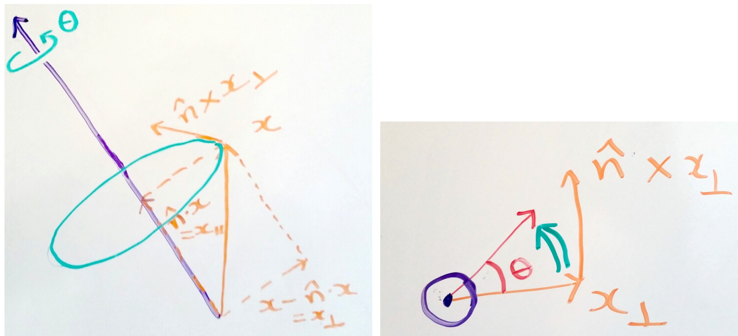

Figure 1: Deriving the axis angle representation of a rotation.

To see that such a point exists, note that:

$$\begin{aligned}
\det(\mathbf{R}-\mathbf{I}) \quad &= \det(\mathbf{R}^T-\mathbf{I}^T) \\
&= \det(\mathbf{R}^T-\mathbf{I}) \\
&= \det(\mathbf{R}^T-\mathbf{R}^T\mathbf{R}) \\
&= \det(\mathbf{R}^T(\mathbf{I}-\mathbf{R})) \\
&= \det(\mathbf{R}^T)\det(\mathbf{I}-\mathbf{R}) \\
&= \det(\mathbf{I}-\mathbf{R}) \\
\Rightarrow \det(\mathbf{R}-\mathbf{I}) = 0 & \qquad (2)
\end{aligned}$$

Thus there exists non-zero $\mathbf{x}$ such that $\mathbf{R}\mathbf{x} = \mathbf{x}$. Alternatively, one of the eigenvalues of $\mathbf{R}$ is 1 and the corresponding eigenvector is the axis. Thus every rotation in 3D amounts to *rotation about an axis by an angle.*

1 Axis-angle representations for a rotation

Let us derive the rotation matrix for a rotation about an axis $\hat{\mathbf{n}}$ by an angle θ (see Figure 1

Consider an arbitrary vector $\mathbf{x}$ in 3D. Then this vector can be broken into two components, one parallel to $\hat{\mathbf{n}}$ and another perpendicular to it. That is,

$$\mathbf{x} = \mathbf{x}_{\parallel} + \mathbf{x}_{\perp} \qquad (3)$$

$$\mathbf{x}_{\parallel} = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{x}) = \hat{\mathbf{n}}\hat{\mathbf{n}}^T\mathbf{x} \qquad (4)$$

$$\mathbf{x}_{\perp} = \mathbf{x} - \mathbf{x}_{\parallel} = (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{x} \qquad (5)$$

A rotation about $\hat{\mathbf{n}}$ will not affect $\mathbf{x}_{\parallel}$, but will affect $\mathbf{x}_{\perp}$. In particular, it will add a new component that is both perpendicular to $\mathbf{x}_{\perp}$ and to the axis $\hat{\mathbf{n}}$. The direction of this component is thus $\hat{\mathbf{n}} \times \mathbf{x}_{\perp} = \hat{\mathbf{n}} \times \mathbf{x}$. The magnitude will depend on θ, in particular, it will be $\sin\theta$. Simultaneously, the rotation will also reduce the component along $\mathbf{x}_\perp$ by a factor of $\cos\theta$. Thus, the rotated point will have the equation:

$$\mathbf{x}' = \mathbf{x}_{\|} + \mathbf{x}_\perp \cos\theta + (\hat{\mathbf{n}} \times \mathbf{x}) \sin\theta \tag{6}$$
$$= \hat{\mathbf{n}}\hat{\mathbf{n}}^T\mathbf{x} + (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T)\mathbf{x}\cos\theta + (\hat{\mathbf{n}} \times \mathbf{x}) \sin\theta \tag{7}$$
$$= \mathbf{x} - (\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T)(1 - \cos\theta)\mathbf{x} + (\hat{\mathbf{n}} \times \mathbf{x}) \sin\theta \tag{8}$$

Finally, note that $\hat{\mathbf{n}} \times \mathbf{x}$ can be written as a matrix multiplication with a special matrix $\hat{\mathbf{n}}_\times = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$. It can also be shown that $\hat{\mathbf{n}}_\times^2 = -(\mathbf{I} - \hat{\mathbf{n}}\hat{\mathbf{n}}^T)$. Thus, we have:

$$\mathbf{x}' = \mathbf{x} + (1 - \cos\theta)\hat{\mathbf{n}}_\times^2\mathbf{x} + \sin\theta\hat{\mathbf{n}}_\times\mathbf{x} \tag{9}$$
$$= \mathbf{R}(\theta, \hat{\mathbf{n}})\mathbf{x} \tag{10}$$

where:

$$\mathbf{R}(\theta, \hat{\mathbf{n}}) = \mathbf{I} + \hat{\mathbf{n}}_\times \sin\theta + \hat{\mathbf{n}}_\times^2(1 - \cos\theta) \tag{11}$$

This equation is called Rodriguez's formula.

Using the Rodriguez' formula, note that

$$\text{trace}(\mathbf{R}) = \text{trace}(\mathbf{I}) + (1 - \cos\theta)\text{trace}(\hat{\mathbf{n}}\hat{\mathbf{n}}^T - \mathbf{I}) \tag{12}$$
$$= 1 + 2\cos\theta \tag{13}$$

This gives us a way to recover the angle also from the rotation matrix.

Another equivalent formula comes from *exponential twists*. We know from physics that the velocity of a point $\mathbf{x}$ rotating with an angular velocity ω about axis $\hat{\mathbf{n}}$ is:

$$\mathbf{v} = \omega\hat{\mathbf{n}} \times \mathbf{x} \tag{14}$$
$$\Rightarrow \frac{d\mathbf{x}}{dt} = \frac{d\theta}{dt}\hat{\mathbf{n}}_\times\mathbf{x} \tag{15}$$
$$\Rightarrow \frac{d\mathbf{x}}{d\theta} = \hat{\mathbf{n}}_\times\mathbf{x} \tag{16}$$

Recall that for scalar variables, an equation of the form $\frac{dx}{d\theta} = ax$ has a solution $x(\theta) = e^{a\theta}x(0)$. Similarly, for this vector equation above, the solution is

$$\mathbf{x}(\theta) = e^{\hat{\mathbf{n}}_\times\theta}\mathbf{x}(0) \tag{17}$$

Here this is the matrix exponential:

$$e^{\mathbf{M}} = \mathbf{I} + \mathbf{M} + \frac{\mathbf{M}^2}{2!} + \frac{\mathbf{M}^3}{3!} + \dots \tag{18}$$

This gives us a second equation

$$\mathbf{R}(\theta, \hat{\mathbf{n}}) = e^{\hat{\mathbf{n}}_\times\theta} \tag{19}$$