May 7, 2020

$$f \sim GP(\mu, k) \qquad X \Rightarrow f_X \sim N(\mu_X, K_{XX})$$

Observations: $y = f_X + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I)$

$$\max_{\sigma} \; \mathcal{L}(\sigma \mid y) = p_\sigma(y) = \frac{\exp\left(-\frac{1}{2} y^T K_\sigma^{-1} y\right)}{\sqrt{(2\pi)^d \det(K_\sigma)}}$$

$$\log \mathcal{L} = -\frac{1}{2} y^T K_\sigma^{-1} y - \frac{d}{2}\log(2\pi) - \frac{1}{2}\log\det(K_\sigma)$$

z

$$\frac{\partial L}{\partial \sigma} y^T K_\sigma^{-1} y = y^T K_\sigma^{-1} \frac{\partial K_\sigma}{\partial \sigma} K_\sigma^{-1} y$$

$2\sigma I$

$$\frac{\partial L}{\partial \sigma} \log\det(K_\sigma) = \text{trace}\left(K_\sigma^{-1} \frac{\partial K_\sigma}{\partial \sigma}\right)$$

$2\sigma I$

- K_σ^{-1} $\quad K_\sigma z = y$ $\quad O(N^3)$
- $\log\det(K_\sigma) = \log\det(U \Lambda U^T) = \log\det(U^T U \Lambda)$ ($U^T U = I$)
  $= \log \prod_i \lambda_i = \sum_i \log(\lambda_i)$
  $O(N^3)$
- $\text{trace}(K_\sigma^{-1} \cdot 2\sigma I)$ $\quad O(N^3)$

$x_i \in \mathbb{R}^2, \mathbb{R}^3$ (spatial data)

$k(x_i, x_j) = \exp(-\|x_i - x_j\|_2^2)$

$O(N)$

$O(N)$

hierarchical low rank

$\mathcal{H}$-matrix

6220

skeletonization

$Ax = b$ $\Leftarrow$ factorization $LUx = b$

iterative

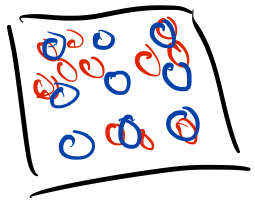
Krylov subspace $\{b, Ab, A^2b, \dots\}$ (6210)

$z^T \underset{\sim}{A} z$

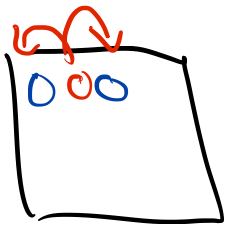
Structured kernel interp (SKI)

$$K_{XX} \approx W K_{U,U} W^T$$

U chosen carefully (uniform on grid)

X 

$$K_{UU} = \begin{bmatrix} a & b & c & d \\ b & a & b & c \\ c & b & a & b \\ d & c & b & a \end{bmatrix}$$



6220: $K_{UU} x$ in $O(N)$

$y = K_{UU} x$

$$(K_{XX})_{ij} = k(x_i, x_j) \approx w_i^T K_{U,U} w_j$$

$$\mathrm{tr}(A) = \sum_i A_{ii}$$

$$z_i = \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$$

$$\mathbb{E}(z^T A z) = \mathrm{tr}(A)$$

$z \sim$ iid Rad

$$\mathbb{E}(z^T A z) = \mathbb{E}\left(\sum_{ij} z_i z_j A_{ij}\right)$$

$$= \sum_{ij} A_{ij} \mathbb{E}(z_i z_j)$$

$$z_i z_j \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases} \quad i \neq j$$

$$\mathbb{E}(z_i z_j) = \mathbb{E}(z_i) = 0$$

$$= \sum_i A_{ii} \mathbb{E}(z_i^2) \quad 1$$

$$z_i^2 = 1 \text{ w.p. } 1$$

$$= \sum_i A_{ii}$$

$$= \mathrm{tr}(A)$$