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Last time: $\mathrm{NCUT}(S) = \dfrac{\mathrm{cut}(S)}{v(S)} + \dfrac{\mathrm{cut}(\bar{S})}{v(\bar{S})}$

$$\mathrm{cut}(S) = \sum_{i \in S, j \in \bar{S}} A_{ij} \qquad v(S) = \sum_{i \in S} d_i$$

$$\phi(S) = \mathrm{cut}(S) / \min(v(S), v(\bar{S}))$$

$$\frac{\mathrm{cut}(S)}{v(S)} + \frac{\mathrm{cut}(\bar{S})}{v(\bar{S})} \le \frac{\mathrm{cut}(S)}{\min(v(S), v(\bar{S}))} + \frac{\mathrm{cut}(\bar{S})}{\min(v(S), v(\bar{S}))}$$

$$= 2\phi(S)$$

$$\ge \frac{\mathrm{cut}(S)}{\min(v(S), v(\bar{S}))} = \phi(S)$$

$$\lambda_2 \le \min_S \mathrm{NCUT}(S)$$

$$\lambda_2 / 2 \le \min_S \phi(S) \le \sqrt{2 \lambda_2}$$

Cheeger

$$\min_x \; x^T L x \quad \text{s.t.} \quad \mathbb{1}^T D x = 0 \quad x^T D x = 1$$

$$z = D^{1/2} x \qquad \min \; z^T N z \qquad N = D^{-1/2} L D^{-1/2}$$

$$\text{s.t.} \quad \mathbb{1}^T D^{1/2} z = 0 \quad z^T z = 1$$

$$N z_* = \lambda_2 z_* \Rightarrow z_*^T N z_* = \lambda_2$$

$$\lambda_2 = \min_{\mathbb{1}^T D^{1/2} z = 0} \frac{z^T N z}{z^T z} \quad \text{at } z_* \qquad \leq \min_S \mathrm{NCUT}(S)$$

$$= \min_{\mathbb{1}^T D x = 0} \frac{x^T L x}{x^T D x} \quad (\text{at } x^* = D^{-1/2} z_*)$$

Let $\underline{\mathbb{1}^T D y} = 0$ $\quad S_t = \{ i \mid y_i < t \}$

$\boxed{\text{Claim}}$: $$\min_t \phi(S_t) \le \sqrt{2\frac{y^T L y}{y^T D y}} \triangleq \sqrt{2R}$$

$$\min_y \frac{y^T L y}{y^T D y} \Rightarrow \lambda_2(N) \qquad y_* = D^{-1/2} z_* \qquad N z_* = \lambda_2 z_*$$

WLOG: $y_1 \le \cdots \le y_n$

$$c = \min_k \sum_{i=1}^{k} d_i \ge v(V)/2 \qquad z \triangleq y - y_c \mathbb{1} \qquad \ast$$

$$\frac{z^T L z}{z^T D z} = \frac{y^T L y}{z^T D z} \le \frac{y^T L y}{y^T D y} = R$$

$$(y + s\mathbb{1})^T D (y + s\mathbb{1}) \Rightarrow 2\mathbb{1}^T D (y + s\mathbb{1}) = 0$$

$z_1^2 + z_n^2 = 1$ (re-scale)

Define prob. dist. on t so that

$$\mathbb{E}(\text{cut}(S_t)) \le \sqrt{2R}\, \mathbb{E}(\min(v(S_t), v(\overline{S_t})))$$

$$\Rightarrow \exists t \quad \text{cut}(S_t) \le \sqrt{2R} \min(v(S_t), v(\overline{S_t}))$$

$$\Rightarrow \phi(S_t) \le \sqrt{2R}$$

$$\text{Prob}(t \in [a,b] \subseteq [z_1, z_n])$$

$$= \int_a^b 2|t|\,dt = \text{sgn}(b)\,b^2 - \text{sgn}(a)\,a^2$$

$$\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Sub-claim: $\mathbb{E}(\min(v(S_t), v(\bar{S}_t))) \geq z^T D z$ $(*)$

$$\mathbb{E}_t(v(S_t)) = \sum_i \Pr(i \in S_t)\, d_i = \sum_i \Pr(z_i \leq t)\, d_i$$

$t < 0 \Rightarrow v(S_t) = \min(v(S_t), v(\bar{S}_t))$

$t \geq 0 \Rightarrow v(\bar{S}_t) =$ ''

$i < c \Rightarrow i$ is in smaller set if $t < 0$

$j \geq c \Rightarrow j$ $\cdot$ $t \geq 0$

$$\mathbb{E}(\min(v(S_t), v(\bar{S}_t))) = \sum_{i<c} \underbrace{\Pr(z_i < t < 0)}_{z_i^2}\, d_i + \sum_{j \geq c} \underbrace{\Pr(0 \leq t \leq z_j)}_{z_j^2}\, d_j = z^T D z$$

Sub-claim (*): $\Pr(i \in S_t, j \in \bar{S}_t) \le |z_i - z_j|(|z_i| + |z_j|)$

$= \Pr(z_i \le t < z_j)$

$= \mathrm{sgn}(z_j) z_j^2 - \mathrm{sgn}(z_i) z_i^2$

$$= \begin{cases} |z_i^2 - z_j^2| & \mathrm{sgn}(z_j) = \mathrm{sgn}(z_i) \\ z_j^2 + z_i^2 & \mathrm{sgn}(z_j) > \mathrm{sgn}(z_i) \end{cases}$$

Exercise:

C-S

$|\langle u,v\rangle|^2 \le \langle u,u\rangle \langle v,v\rangle$

$$\mathbb{E}(\mathrm{cut}(S_t)) \overset{*}{\le} \sum_{(i,j)\in E} |z_i - z_j|\,(|z_i| + |z_j|)$$

$$\le \sqrt{\sum_{(i,j)\in E} (z_i - z_j)^2}\,\sqrt{\sum_{(i,j)\in E} (|z_i| + |z_j|)^2}$$

$$\overset{*}{\le} \sqrt{R\, z^T D z}\,\sqrt{\sum_{(i,j)\in E} 2(z_i^2 + z_j^2)}$$

$$= \sqrt{R}\,\sqrt{z^T D z}\,\sqrt{2}\,\sqrt{z^T D z}$$

$$= \sqrt{2R}\, z^T D z$$

$$\overset{*}{=} \sqrt{2R}\, \mathbb{E}\left(\min(v(S_t), v(\bar{S}_t))\right)$$

Alg: $G = (V, E)$

$N = D^{-1/2} L D^{-1/2}$

$N v_2 = \lambda_2 v_2$

$x = D^{-1/2} v_2$

σ = sort x by value

for $t = 1, \ldots, n$

$$\phi_t = \phi(\overbrace{\{\sigma_1, \ldots, \sigma_t\}}^{S_t})$$

return S_t that minimizes cond

guarantee: $\phi(S_t) \underbrace{\leq \sqrt{2\lambda_2}}_{\text{upper bound}} \leq \underbrace{\sqrt{2 \cdot 2 \phi_*}}_{\text{lower bound}} \leq 2\sqrt{\phi_*}$

$\lambda_2/2 \leq \phi_* \leq \sqrt{2\lambda_2}$

k-way NCUT

$$NCUT(S_1, \ldots, S_k) = \sum_{r=1}^{k} \frac{cut(S_r)}{v(S_r)}$$

test matrix $\quad n \; \overset{k}{\boxed{X}}$

$$X_{ir} = \begin{cases} 1/\sqrt{v(S_r)} & i \in S_r \\ 0 & o/w \end{cases}$$

Claim: $tr(X^T L X) = NCUT \qquad tr(M) = \sum_i M_{ii}$

$$= \sum_{r=1}^{k} x_r^T L x_r$$

$$x_r^T L x_r = \sum_{(i,j)\in E} (x_{ir} - x_{jr})^2$$

$$= \sum_{i\in S_r,\, j\in \bar{S}_r} \left(\frac{1}{\sqrt{v(S_r)}} - 0\right)^2$$

Claim: $X^T D X = I$

Proof: (r, s) $e_r^T X^T D X e_s = x_r^T D x_s = \sum_i d_i x_{ir} x_{is}$

$$= \sum_i d_i x_{ir}^2$$

$$= \sum_{i \in S_r} d_i \frac{1}{v(S_r)} = 1$$

$$\min_{S_1,\dots,S_k} \operatorname{tr}(X^T L X) = \text{NCUT}$$

$$\text{s.t.} \quad X^T D X = I \qquad \cancel{X_{ir} = \begin{cases} 1/\sqrt{\mathrm{vol}(S_r)} & i \in S_r \\ 0 & \text{o/w} \end{cases}}$$

$$Z = D^{1/2} X$$

$$\min_{Z} \operatorname{tr}(Z^T N Z) \quad \text{s.t.} \quad Z^T Z = I$$

$$Z = \underbrace{V(:,1:k)}_{V_k} \qquad (\text{Courant-Fischer})$$

$$N V_k = \Lambda_k V_k \qquad (k \text{ smallest evals})$$

$$X_* = D^{-1/2} V_k$$