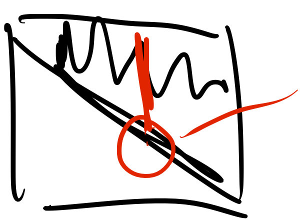
Feb 20, 2020

$\boxed{A}\,\boxed{\Pi} = \boxed{Q}\,R \qquad Q^T Q = I$

Pivoted QR

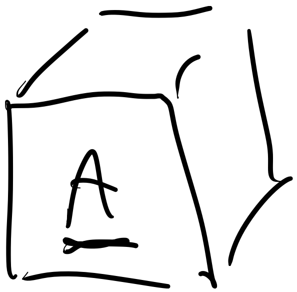
1. Choose column t with largest 2-norm
2. Pivot: Swap $A(:,1)$ and $A(:,t)$
3. $q_1 = A(:,1) / \|A(:,1)\|_2$
4. $A \longleftarrow (I - q_1 q_1^T) A = A - q_1 \underbrace{q_1^T A}_{= R(1,:)}$
5. Repeat on $A(:,2:n)$

R  ≈ 0

$|r_{1,j}| \le |r_{1,1}|$

$|r_{2,2}| \le |r_{1,1}|$

Last time: tensors



low-rank approx

$$\min_{\underline{B}} \|\underline{A} - \underline{B}\|_F^2 \quad \text{s.t. } \underline{B} \text{ has rank-}r$$

$$\underline{B} = \sum_{s=1}^{r} x_s \otimes y_s \otimes z_s$$

Problem: can be ill-posed

Other problems:

(1) no nesting of best orthog. low rank-approx

Matrix: $\min_B \|A - B\|$ s.t. B rank r $r+1$

$$B_r = U_r \Sigma_r V_r^T = \sum_{s=1}^{r} \sigma_s u_s \otimes v_s$$

$$B_{r+1} = B_r + \sigma_{r+1} u_{r+1} \otimes v_{r+1}$$

cannot have this for tensors (Kolda 03)

(2) Computing the rank of tensor is NP-hard (Håstad 90)

(3) Symm. tensors $\underline{A}$ $\quad \underline{A}_{ijk} = \underline{A}_{ikj} = \underline{A}_{jik} = \underline{A}_{jki}$

$= \underline{A}_{kij} = \underline{A}_{kji}$

Best symm. rank-1 approx.

$$\min_{\gamma, x} \|\underline{A} - \gamma\, x \otimes x \otimes x\|_F^2 \qquad \|x\|_2 = 1$$

Still NP-hard

(4) allow non-symm.

$$\min_{\gamma, x, y, z} \|\underline{A} - \gamma\, x \otimes y \otimes z\|_F^2 \quad \text{s.t. } \|x\|_2 = \|y\|_2 = \|z\|_2 = 1$$

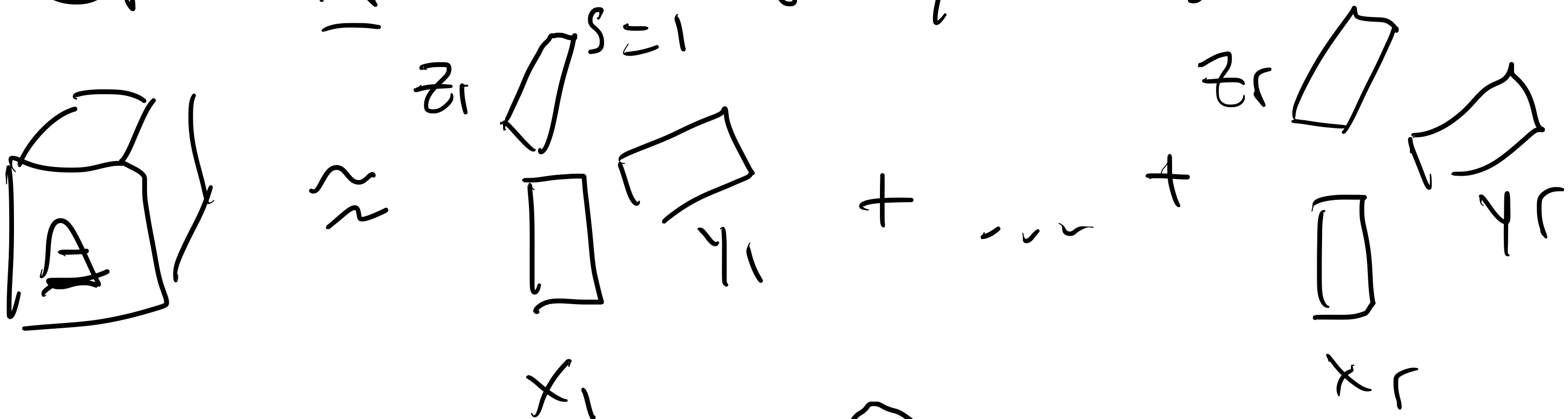
Equiv. (Banach 30)

"Most tensor problems are NP-hard" (Hillar-Lim 2013)

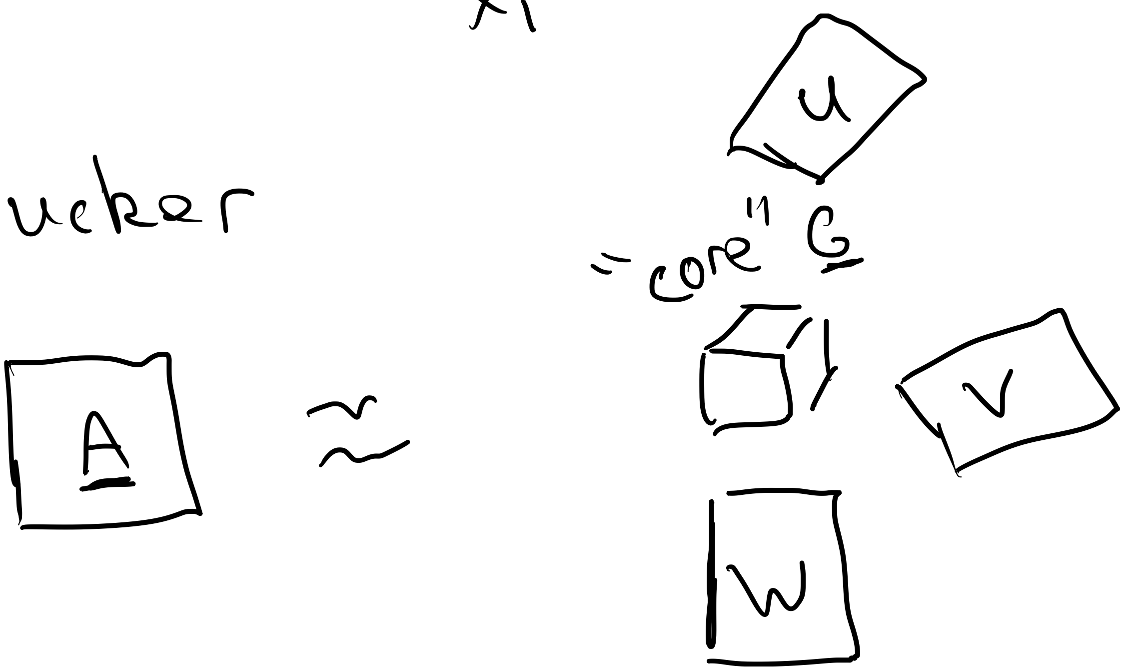
1. Approx / heuristics
2. Special cases / constrain the problem
3. regularization (for ill-posedness)

Two types of tensor decompositions

① CP $\underline{A} \approx \sum_{s=1}^{r} x_s \otimes y_s \otimes z_s$



② Tucker



$U^T U = I$

$V^T V = I$

$W^T W = I$

CP decomposition

CANDECOMP canonical decomposition

PARAFAC Parallel factors

Canonical polyadic

$$\min_{X,Y,Z} \left\| \underline{A} - \sum_{s=1}^{r} x_s \otimes y_s \otimes z_s \right\|_F^2 \qquad X = [x_1 \cdots x_r]$$

Alternating least squares: fix rank

1. Fix Y, Z, solve X
2. Fix X, Z, solve Y
3. Fix X, Y solve for Z

$$\min_X \left\| \underline{A} - \sum_{s=1}^{r} x_s \otimes y_s \otimes z_s \right\|_F^2$$

$$= \sum_{ijk} \left(\underline{A}_{ijk} - \sum_{s=1}^{r} x_s(i) \cdot y_s(j) \cdot z_s(k) \right)^2$$

Fix i

pick some jk

$$\left(\underline{A}_{ijk} - \left[y_1(j) z_1(k) \cdots\cdots y_r(j) z_r(k) \right] \begin{bmatrix} x_1(i) \\ \vdots \\ x_r(i) \end{bmatrix} \right)^2$$

$$\left\| \begin{bmatrix} | \\ \underline{A}_{ijk} \\ | \end{bmatrix}_{jk} - \begin{bmatrix} & | & \\ y_1(j) z_1(k) & \cdots & y_r(j) z_r(k) \\ & | & \end{bmatrix}_{jk} \begin{bmatrix} x_1(i) \\ \vdots \\ x_r(i) \end{bmatrix} \right\|_2^2$$

LLS for ith row of $\boxed{X}_i$

Khatri-Rao product $Z \odot Y$

$$\left\| \underline{A}(i,:,:) - (Z \odot Y)\, X(i,:) \right\|_2^2 \Rightarrow \left\| A_{(1)} - (Z \odot Y) X^T \right\|_F^2$$

① $X^{k+1} = \underset{X}{\arg\min} \| \underline{A}_{(1)} - (Z^k \odot Y^k) X^T \|_F^2$

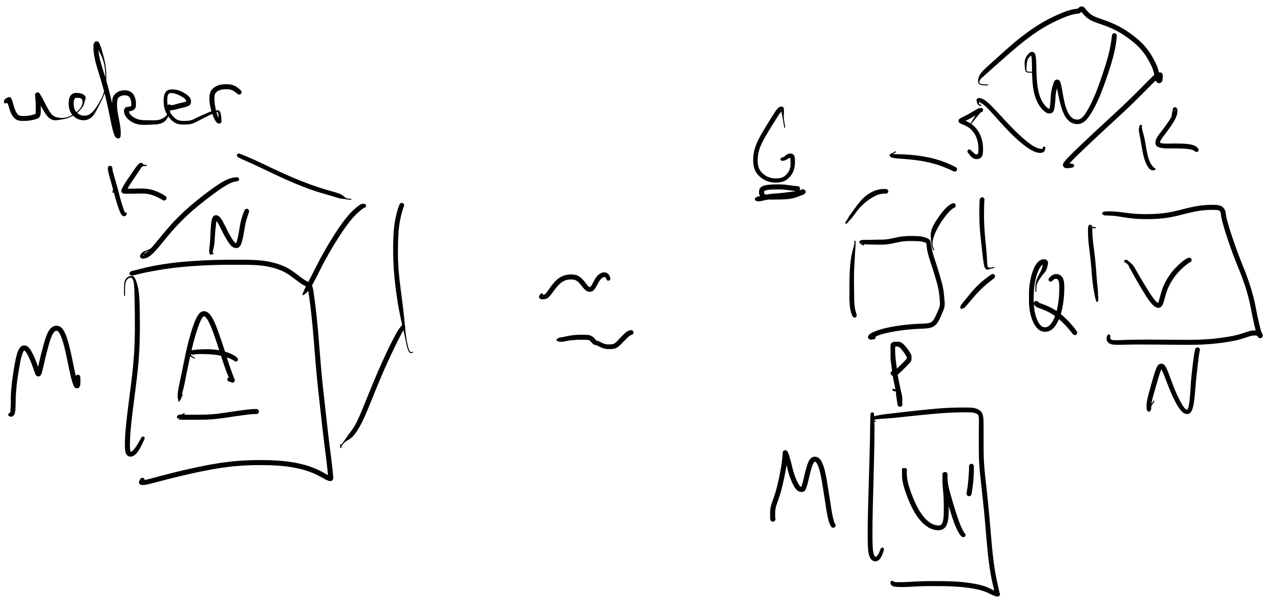
② $Y^{k+1} = \underset{Y}{\arg\min} \| \underline{A}_{(2)} - (Z^k \odot X^{k+1}) Y^T \|_F^2$

③ $Z^{k+1} = \underset{Z}{\arg\min} \| \underline{A}_{(3)} - (Y^{k+1} \odot X^{k+1}) Z^T \|_F^2$

Stop if error $\leq \varepsilon$ or niter $< N$

regularization $\lambda (\|X\|_F^2 + \|Y\|_F^2 + \|Z\|_F^2)$

Tucker



$U^TU = I$

$V^TV = I$

$W^TW = I$

$\underline{G}$ is $P \times Q \times S$

$$\underline{A}_{ijk} \approx \sum_{p=1}^{P} \sum_{q=1}^{Q} \sum_{s=1}^{S} \underline{G}_{pqs}\, u_p \otimes v_q \otimes w_s$$

$$\underline{G}_{pqs} = \begin{cases} 1 & p=q=s \\ 0 & o/w \end{cases}$$

like CP

Assume we know U, V, W

$$\min_{\underline{G}} \| \underline{A} - \sum_{pqs} \underline{G}_{pqs}\, u_p \otimes v_q \otimes w_s \|_F^2$$

$$\Rightarrow \text{LLS} \quad \underline{G}^* = f(\underline{A}, U, V, W)$$

Can show:

$$\min_{\underline{G}, U, V, W} \| \underline{A} - \sum_{pqs} \underline{G}_{pqs}\, u_p \otimes v_q \otimes w_s \|_F^2$$

equiv. to $\max_{\underline{G}, U, V, W} \| \underline{G} \|_F^2 \quad U^T U = I \quad V^T V = I \quad W^T W = I$

s.t. $\underline{G} = f(\underline{A}, U, V, W)$

$\underline{G}^k = f(\underline{A}, U^k, V^k, W^k)$

fix V^k, W^k

$$\max_{\underline{G}, U} \|\underline{G}\| \quad \text{s.t.} \quad \underline{G} = f(\underline{A}, U, V^k, W^k)$$

$$\max_{U} \|U^T B\|_F^2 \quad B = A_{(1)}(V^k \otimes W^k)$$

Kolda & Bader 09