Homework 1, CS 6241 Spring 2020

Instructor: Austin R. Benson
Due Thursday, Feburary 13, 2020 at 11:59pm ET on CMS

Theoretical analysis

In the first 4 lectures, we covered least square solutions

$$\min_x \|Ax - b\|_2^2, \tag{1}$$

where A was $m \times n$ with $m > n$ with full rank n. This is usually called *overdetermined* since there are more equations than unknowns. In this part of the assignment, we examine the *underdetermined* case, where $m < n$. We will assume that A has full rank m.

Since A has rank m, we can certainly find an x such that $Ax = b$, resulting in zero least squares error. However, there are now infinitely many such x. If we have one particular solution $Ax_0 = b$, then all solutions are given by combining x_0 with the null space of A:[1]

$$Z = \{x_0 + y \mid Ay = 0\} = \arg\min_x \|Ax - b\|_2^2. \tag{2}$$

[1] If you haven't seen this before, try proving it on your own.

When we talked about regularization in class, we wanted to enforce some structure on the solution. We can do that in the underdetermined case, too. Here, we think about the smallest or *least-norm* solution:

$$x_{\text{ln}} = \arg\min_{z \in Z} \|z\|_2. \tag{3}$$

1. Show that the least-norm solution is $x_{\text{ln}} = A^T \left(AA^T\right)^{-1} b$.[2]

[2] Hint: use the characterization of the solution set in Equation (2).

2. The least-norm solution is related to the Tikhonov regularization we used in class for encouraging small solutions:

$$x_\lambda = \arg\min_x \left\{ \|Ax - b\|_2^2 + \lambda^2 \|x\|_2^2 \right\} \tag{4}$$

Convince yourself that x_λ is unique[3] and then show that

[3] Hint: can we turn this into an overdetermined least-squares problem?

$$\lim_{\lambda \to 0} x_\lambda = x_{\text{ln}}. \tag{5}$$

3. The least-norm solution can also be found by gradient descent using the objective function $f(x) = \|Ax - b\|_2^2$. Suppose that our initial guess is $x_0 = A^T y$ for arbitrary y. Show that if gradient descent converges to a minimizer x_*, then $x_* = x_{\text{ln}}$. Recall that the gradient descent iterations are

$$x_{k+1} = x_k - \alpha_k \nabla f(x_k). \tag{6}$$

Data analysis

Find a real-world dataset online[4] or from your own work / research and set up a linear least squares problem. Write code to solve the standard least squares problem and also one of the regularized least squares problems from class (or implement another regularization technique that you find on your own). You are permitted to re-use code from the course Jupyter notebooks for this part of the assignment.[5]

Provide a brief description of the dataset, the regression problem, and why the regularization technique you chose might be useful. Also provide a brief qualitative analysis of how the solution changes with the regularization.

[4] The course web site (`https://www.cs.cornell.edu/courses/cs6241/2020sp/`) has some pointers for datasets.

[5] `https://github.com/arbenson/cs6241_2020sp`

Preparation & submission guidelines

Typesetting. All homeworks should be prepared with LaTeX. Handwritten homeworks will not be accepted.

Code. Part of the assignment involves writing code. You need to include your code in your submission, and you can easily do so using the `listings` package. You do not need to include code that you write for the qualitative analysis.

Collaboration. You are encouraged to discuss and collaborate on the homework. However, you have to write up your own solutions and write your own code. You must also list your collaborators on your homework.

Academic Integrity. We expect you to maintain academic integrity in the course. For example, follow the collaboration guidelines above and do not just copy someone else's code. Failure to maintain academic integrity will be penalized severely. Plagiarism is a form of academic misconduct, so make sure to provide proper citations. Cornell has a number of guidelines on plagiarism.[6]

[6] https://plagiarism.arts.cornell.edu/tutorial/index.cfm

Submission. Your homework should be submitted as a single PDF that includes your solutions to the theoretical analysis and data analysis components, along with your code (as outlined above). Also include your name and the names of any collaborators. Submit your PDF on CMS.[7]

[7] https://cmsx.cs.cornell.edu