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# Eigenvalues of a real supersymmetric tensor

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**Abstract**

In this paper, we define the symmetric hyperdeterminant, eigenvalues and E-eigenvalues of a real supersymmetric tensor. We show that eigenvalues are roots of a one-dimensional polynomial, and when the order of the tensor is even, E-eigenvalues are roots of another one-dimensional polynomial. These two one-dimensional polynomials are associated with the symmetric hyperdeterminant. We call them the characteristic polynomial and the E-characteristic polynomial of that supersymmetric tensor. Real eigenvalues (E-eigenvalues) with real eigenvectors (E-eigenvectors) are called H-eigenvalues (Z-eigenvalues). When the order of the supersymmetric tensor is even, H-eigenvalues (Z-eigenvalues) exist and the supersymmetric tensor is positive definite if and only if all of its H-eigenvalues (Z-eigenvalues) are positive. An $m$th-order $n$-dimensional supersymmetric tensor where $m$ is even has exactly $n(m-1)^{n-1}$ eigenvalues, and the number of its E-eigenvalues is strictly less than $n(m-1)^{n-1}$ when $m \geq 4$. We show that the product of all the eigenvalues is equal to the value of the symmetric hyperdeterminant, while the sum of all the eigenvalues is equal to the sum of the diagonal elements of that supersymmetric tensor, multiplied by $(m-1)^{n-1}$. The $n(m-1)^{n-1}$ eigenvalues are distributed in $n$ disks in $\mathbf{C}$. The centers and radii of these $n$ disks are the diagonal elements, and the sums of the absolute values of the corresponding off-diagonal elements, of that supersymmetric tensor. On the other hand, E-eigenvalues are invariant under orthogonal transformations.



*Keywords:* Eigenvalue; Supersymmetric tensor; Symmetric hyperdeterminant

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## 1. Introduction

A real $m$th-order $n$-dimensional tensor $A$ consists of $n^m$ real entries:

$$A_{i_1,\ldots,i_m} \in \mathbf{R},$$

where $i_j = 1, \ldots, n$ for $j = 1, \ldots, m$. The tensor $A$ is called supersymmetric if its entries are invariant under any permutation of their indices (Kofidis and Regalia, 2002).

The tensor $A$ defines an $m$th-degree homogeneous polynomial $f(x) \in \mathbf{R}[x_1, \ldots, x_n]$, $x = (x_1, \ldots, x_n)$:

$$f(x) \equiv Ax^m := \sum_{i_1,\ldots,i_m=1}^{n} A_{i_1,\ldots,i_m} x_{i_1} \cdots x_{i_m}, \tag{1}$$

where $x^m$ can be regarded as an $m$th-order $n$-dimensional rank-one tensor with entries $x_{i_1} \cdots x_{i_m}$ (Kofidis and Regalia, 2002), and $Ax^m$ is the tensor product of $A$ and $x^m$. Clearly, if $A$ is not supersymmetric, we may replace $A$ by a supersymmetric tensor $\bar{A}$ such that

$$f(x) \equiv \bar{A}x^m \equiv Ax^m.$$

We denote this supersymmetric tensor $\bar{A}$ as $\mathrm{sym}(A)$.

In 1845, Cayley initiated the study of hyperdeterminants (Cayley, 1845). It was assumed that hyperdeterminants would play a role for tensors like determinants for matrices. But this study was largely abandoned for 150 years until the book (Gelfand et al., 1994) appeared.

Recently, motivated by the study of positive definiteness of $f(x)$ defined in (1), Qi (2004) introduced the concepts of H-eigenvalues and Z-eigenvalues of an even-order real supersymmetric tensor $A$.

When $m$ is even, the positive definiteness of such a homogeneous polynomial form $f(x)$ plays an important role in the stability study of nonlinear autonomous systems via Lyapunov's direct method in automatic control (Anderson et al., 1975; Bose and Kamt, 1974; Bose and Newcomb, 1974; Hsu and Meyer, 1968). We say that a supersymmetric tensor $A$ is positive definite if $f(x)$ defined by (1) is positive definite. Researchers in automatic control studied the conditions of such positive definiteness intensively (Anderson et al., 1975; Bose and Kamt, 1974; Bose and Modaress, 1976; Bose and Newcomb, 1974; Fu, 1998; Hasan and Hasan, 1996; Hsu and Meyer, 1968; Jury and Mansour, 1981; Ku, 1965; Wang and Qi, 2005). For $n \le 3$, the positive definiteness of such a homogeneous polynomial form can be checked by a method based on the Sturm theorem (Bose and Modaress, 1976). For $n \ge 3$ and $m \ge 4$, this issue is a hard problem in mathematics.

For a vector $x \in \mathbf{R}^n$, we use $x_i$ to denote its components, and $x^{[m]}$ to denote a vector in $\mathbf{R}^n$ such that

$$x_i^{[m]} = x_i^m$$

for all $i$. By the tensor product (Qi and Teo, 2003), $Ax^{m-1}$ for a vector $x \in \mathbf{R}^n$ denotes a vector in $\mathbf{R}^n$, whose $i$th component is

$$\sum_{i_2,\ldots,i_m=1}^{n} A_{i,i_2,\ldots,i_m} x_{i_2} \cdots x_{i_m}.$$

Qi (2004) called a real number $\lambda$ an **H-eigenvalue** of $A$ if it and a nonzero real vector $x$ are solutions of the following homogeneous polynomial equation:

$$Ax^{m-1} = \lambda x^{[m-1]}, \tag{2}$$

and called the solution $x$ an **H-eigenvector** of $A$ associated with the H-eigenvalue $\lambda$. Qi (2004) also called a real number $\lambda$ and a real vector $x \in \mathbf{R}^n$ a **Z-eigenvalue** of $A$ and a **Z-eigenvector** of $A$ associated with the Z-eigenvalue $\lambda$ respectively, if they are solutions of the following system:

$$\begin{cases} Ax^{m-1} = \lambda x \\ x^{\mathrm{T}} x = 1. \end{cases} \tag{3}$$

It was proved in Qi (2004) that H-eigenvalues and Z-eigenvalues exist for an even-order real supersymmetric tensor $A$, and $A$ is positive definite if and only if all of its H-eigenvalues (Z-eigenvalues) are positive. Thus, the smallest H-eigenvalue and the smallest Z-eigenvalue of an even-order supersymmetric tensor $A$ are important indicators of positive definiteness of $A$. When $n$ is very small, we may use (2) and (3) to calculate all H-eigenvalues (Z-eigenvalues) of $A$, then judge whether $A$ is positive definite or not. In general, Qi (2004) gave several computable upper and lower bounds of the smallest Z-eigenvalue and H-eigenvalue of $A$, and presented a procedure for improving these upper bounds.

For a supersymmetric tensor $A$, we define its symmetric hyperdeterminant, denoted by $\det(A)$, as an irreducible polynomial in $A_{i_1,\ldots,i_m}$, which vanishes wherever there is an $x \in C^n, x \neq 0$, such that $f(x) = 0$ and its gradient $\nabla f(x) = 0$. Note that when $m = 2$ this definition coincides with that of the determinant of a symmetric matrix, but in general it is different from the hyperdeterminant introduced by Cayley. The symmetric hyperdeterminant of $A$ is actually the resultant of the system $\nabla f(x) = 0$. As the theory of the resultant (Cox et al., 1998; D'Andrea and Dickenstein, 2001; Gelfand et al., 1994; Sturmfels, 2002) becomes more developed, this definition becomes more usable, as shown in our paper.

We extend the Kronecker symbol to the case of $m$ indices:

$$\delta_{i_1,\ldots,i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

We call an $m$th-order $n$-dimensional tensor the $m$th **unit tensor** if its entries are $\delta_{i_1,\ldots,i_m}$ for $i_1, \ldots, i_m = 1, \ldots, n$, and denote it by $I$. To specify the sign and scale of the symmetric hyperdeterminant, we may let $\det(I) = 1$. Suppose that $m$ is even. It was observed in Qi (2004) that the H-eigenvalues of $A$ are roots of the following one-dimensional polynomial of $\lambda$:

$$\phi(\lambda) = \det(A - \lambda I). \tag{4}$$

The one-dimensional polynomial $\phi$ was called the **characteristic polynomial** of $A$. Qi (2004) attributed "Z-eigenvalues" to Zhou (2004) as Zhou (2004) suggested to the author the definition (3).

The discussion of H-eigenvalues and Z-eigenvalues is restricted for real numbers with real eigenvectors. This is because of the need for discussing the positive definiteness. When $m = 2$, this restriction is unnecessary, as a real symmetric matrix has only real eigenvalues with real eigenvectors. This does not extend to the high order cases. This restriction obstructs the view of the full mathematical structure of eigenvalues of a supersymmetric tensor.

The behaviours of H-eigenvalues are closer to those of eigenvalues of matrices in a certain sense. For example, the H-eigenvalues of a diagonal even-order real supersymmetric tensor are exactly its diagonal elements. The H-eigenvalues have a Gershgorin-type theorem. These two properties do not hold for Z-eigenvalues.

In this paper, we extend H-eigenvalues and Z-eigenvalues to the complex case. This enables us to know the full mathematical structure of eigenvalues of a supersymmetric tensor.

Throughout this paper, we assume that $m, n \geq 2$, and $A$ is an $m$th-order $n$-dimensional real supersymmetric tensor. In the next section, we discuss some properties of the symmetric hyperdeterminant. While most of them can be easily derived from the contents of Gelfand et al. (1994), the proof of Proposition 4 is nontrivial, and it relies on the theory of the resultant (Cox et al., 1998). Proposition 4 is critical for the discussion in Section 3.

Since the behaviours of H-eigenvalues are closer to eigenvalues of matrices in a certain sense, we call a number $\lambda \in \mathbf{C}$ an **eigenvalue** of $A$ if it and a nonzero vector $x \in \mathbf{C}^n$ are solutions of the homogeneous polynomial equation (2), and we call the solution $x$ an **eigenvector** of $A$ associated with the eigenvalue $\lambda$. On the other hand, since the definition (3) is associated with the Euclidean norm, we call a number $\lambda \in \mathbf{C}$ an **E-eigenvalue** of $A$ if it and a nonzero vector $x \in \mathbf{C}^n$ are solutions of the polynomial equation system (3), and we call the solution $x$ an **E-eigenvector** of $A$ associated with the eigenvalue $\lambda$.

In Section 3, we show that a number in $\mathbf{C}$ is an eigenvalue of $A$ if and only if it is a root of the characteristic polynomial $\phi$. We show that $A$ has exactly $n(m-1)^{n-1}$ eigenvalues, the product of all the eigenvalues of $A$ is equal to $\det(A)$, and the sum of all the eigenvalues of $A$ is

$$(m-1)^{n-1}$$

times the sum of the diagonal elements of $A$. We show that when $m$ is even, an E-eigenvalue of $A$ is a root of another one-dimensional polynomial associated with $A$. We call that one-dimensional polynomial the E-characteristic polynomial of $A$. We show that when $m \geq 4$, the number of E-eigenvalues of $A$, counted with multiplicity, is strictly less than $n(m-1)^{n-1}$.

In Section 4, we give a formula for calculating an eigenvalue $\lambda$ using its eigenvector $x$ if $\sum_{j=1}^n x_j^m \neq 0$, and a formula for calculating an E-eigenvalue $\lambda$ using its E-eigenvector $x$ if $\sum_{j=1}^n x_j^2 \neq 0$. We prove that two eigenvectors $x$ and $y$ associated with two distinct eigenvalues $\lambda$ and $\mu$ are linearly independent. When $m$ is even, we prove that $A$ has at least two distinct H-eigenvalues if $A$ is not a multiple of $I$. We also prove there that when $m$ is even, a necessary condition for positive semidefiniteness of $A$ is that $\det(A) \geq 0$.

In Section 5, we study the distribution of eigenvalues and H-eigenvalues. We show that eigenvalues are distributed in $n$ disks in $\mathbf{C}$. The centers and radii of these $n$ disks are the diagonal elements, and the sums of the absolute values of the corresponding off-diagonal elements, of $A$. When $m$ is even, the largest (smallest) H-eigenvalue is always in the rightmost (leftmost) component of the union of the $n$ intervals intersected by these $n$ disks with the real axis. This gives a lower bound and a new upper bound for the smallest H-eigenvalue, which is useful in judging the positive definiteness of $A$ (Qi, 2004). We give an example for $m = 4$ and $n = 3$ there for judging the positive definiteness of $A$ and constructing a formula for $\det(A)$ by calculating all the eigenvalues of $A$.

We prove that E-eigenvalues are invariant under orthogonal transformation in Section 6. Some concluding remarks are given in Section 7.

## 2. Properties of the symmetric hyperdeterminant

We now summarize some properties of the symmetric hyperdeterminants of $A$.

**Proposition 1.** *The symmetric hyperdeterminant of $A$, $\det(A)$, is the resultant of*

$$Ax^{m-1} = 0,$$

*and is a homogeneous polynomial in the entries of $A$, with the degree $d = n(m-1)^{n-1}$. The degree of $A_{i,\ldots,i}$ in $\det(A)$ is not greater than $(m-1)^{n-1}$.*

**Proof.** According to our definition, $\det(A)$ is the resultant of $f(x)$ and $\nabla f(x)$, where $f$ is defined by (1). Since $A$ is supersymmetric,

$$\nabla f(x) \equiv mAx^{m-1}.$$

We see that $f(x) = 0$ if $Ax^{m-1} = 0$. Hence, $\det(A)$ is the resultant of $Ax^{m-1} = 0$. The second and the third conclusions now follow from Proposition 1.1 of Chapter 13 of Gelfand et al. (1994), and the fact that $A_{i,\ldots,i}$ only occurs in the $i$th equation of $Ax^{m-1} = 0$. $\square$

**Corollary 1.** *For any real number $a$,*

$$\det(aA) = a^d \det(A),$$

*where $d = n(m-1)^{n-1}$.*

**Proposition 2.** *If we permute some indices of $A$, the value of its symmetric hyperdeterminant will be invariant.*

**Proof.** This follows from the supersymmetry of $A$ and our definition of the symmetric hyperdeterminant. $\square$

**Proposition 3.** *If $A$ is diagonal, then*

$$\det(A) = \prod_{i=1}^{n} A_{i,\ldots,i}^{(m-1)^{n-1}}.$$

*In general, this will be a term of* $\det(A)$.

**Proof.** Assume that $A$ is diagonal. By our definition, $\det(A)$ should be proportional to the product of the powers of its diagonal elements. By Proposition 2, the degree of each diagonal element in this product should be the same. By Proposition 1, the degree of each diagonal element in this product should be $(m-1)^{n-1}$. Since $\det(I) = 1$, the coefficient of the product is 1. The first conclusion follows. Since the formula of $\det(A)$ when $A$ is diagonal can be obtained by letting all the off-diagonal elements be zero in the formula of $\det(A)$ in the general case while by Proposition 1 $\det(A)$ is a homogeneous polynomial in the entries of $A$ in the general case, the second conclusion follows. $\square$

**Proposition 4.** *In* $\det(A)$, *except for the term*

$$\prod_{i=1}^{n} A_{i,\ldots,i}^{(m-1)^{n-1}}$$

*as stated in Proposition 3, the total degree with respect to $A_{1,\ldots,1}, A_{2,\ldots,2}, \ldots, A_{n,\ldots,n}$ is not greater than*

$$n(m-1)^{n-1} - 2.$$

**Proof.** Denote $F(x) := Ax^{m-1}$.

Suppose that the conclusion is not true. Then by Proposition 1 and Proposition 1.1 of Chapter 13 of Gelfand et al. (1994), without loss of generality, we may assume that in $\det(A)$, there is a term

$$c \prod_{i=1}^{n-1} A_{i,\ldots,i}^{(m-1)^{n-1}} A_{n,\ldots,n}^{(m-1)^{n-1}-1} A_{n,i_2,\ldots,i_m}, \tag{5}$$

where $\delta_{n,i_2,\ldots,i_m} = 0$, and $c$ is a nonzero real number.

In the following, we need the knowledge on resultants in Section 4, Chapter 3, of Cox et al. (1998).

Let $\bar{d} = n(m-1) - n + 1$. For $x = (x_1, \ldots, x_n)$, let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbf{R}[x_1, \ldots, x_n]$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_1, \ldots, \alpha_n$ are nonnegative integers. Denote $|\alpha| = \sum_{i=1}^n \alpha_i$.

Let

$$S = \{x^\alpha : |\alpha| = \bar{d}\}, \quad N = |S|,$$
$$S_1 = \{x^\alpha \in S : x_1^{m-1} \text{ divides } x^\alpha\},$$
$$S_2 = \{x^\alpha \in S \setminus S_1 : x_2^{m-1} \text{ divides } x^\alpha\},$$
$$\ldots$$
$$S_n = \{x^\alpha \in S \setminus \cup_{i=1}^{n-1} S_i : x_n^{m-1} \text{ divides } x^\alpha\}.$$

By Section 4, Chapter 3, of Cox et al. (1998), $\{S_1, \ldots, S_n\}$ is a partition of $S$.

Consider the equations

$$x^\alpha / x_i^{m-1} \cdot F_i = 0, \text{ for all } x^\alpha \in S_i,$$

for $i = 1, \ldots, n$. Regarding the monomials of total degree $\bar{d}$ as unknowns, we get a system of $N$ linear equations in $N$ unknowns. Denote its coefficient matrix by $M_n$ and let $D_n = \det(M_n)$.

Let $x^\alpha \in S$. Then there is a unique $i$ such that $x^\alpha \in S_i$. Then $A_{i,\ldots,i}$ is the coefficient of $x^\alpha$ in $x^\alpha / x_i^{m-1} \cdot F_i = 0$. Thus, by some permutation, we may let the diagonal elements of $M_n$ be the diagonal elements of $A$ while the off-diagonal elements of $M_n$ do not involve diagonal elements of $A$.

Hence the term of $D_n$ which has the highest total degree with respect to $A_{1,\ldots,1}$, $A_{2,\ldots,2}, \ldots, A_{n,\ldots,n}$ is

$$\prod_{i=1}^{n} A_{i,\ldots,i}^{|S_i|}. \tag{6}$$

By Exercise 4.3 in Chapter 3 of Cox et al. (1998), $|S_n| = (m-1)^{n-1}$. By Proposition 4.6 in Chapter 3 of Cox et al. (1998) as well as supersymmetry of $A$,

$$D_n = \det(A) \cdot h, \tag{7}$$

where $h$ is an extraneous factor, which is a polynomial with coefficients $A_{i_1,\ldots,i_m}$, $1 \le i_j \le n-1$ for $j = 1, \ldots, m$. Let $h_0$ be the monomial of $h$, which has the highest total degree of $A_{1,\ldots,1}, \ldots, A_{n-1,\ldots,n-1}$. By Propositions 1 and 3 as well as (6) and (7),

$$\prod_{i=1}^{n} A_{i,\ldots,i}^{|S_i|} = h_0 \prod_{i=1}^{n} A_{i,\ldots,i}^{(m-1)^{n-1}},$$

i.e.,

$$h_0 = \prod_{i=1}^{n-1} A_{i,\ldots,i}^{|S_i|-(m-1)^{n-1}}. \tag{8}$$

In (7), the product of (5) and $h_0$ is

$$c \prod_{i=1}^{n-1} A_{i,\ldots,i}^{|S_i|} A_{n,\ldots,n}^{|S_n|-1} A_{n,i_2,\ldots,i_m}. \tag{9}$$

The total degree of $A_{1,\ldots,1}, \ldots, A_{n,\ldots,n}$ of this term is $N-1$.

Now, suppose the product of a term in $\det(A)$ and a term in $h$ is proportional to (9). Since a term of $\det(A)$ with the highest total degree of $A_{1,\ldots,1}, \ldots, A_{n-1,\ldots,n-1}$ can have the factor

$$\prod_{i=1}^{n-1} A_{i,\ldots,i}^{(m-1)^{n-1}}$$

and $h_0$ is the term in $h$ with the highest total degree of $A_{1,\ldots,1}, \ldots, A_{n-1,\ldots,n-1}$ as shown in (8), comparing with (9), the term in $h$ must be $h_0$. Since $h$ and $h_0$ does not involve

$A_{i_1,\dots,i_m}$ with at least one of $\{i_1, \dots, i_m\}$ equal to $n$, the term in $\det(A)$ must be the term (5). This implies that in $D_n$ expressed as the product of $\det(A)$ and $h$ as in (7), the term (9) cannot be canceled by other products of terms of $\det(A)$ and $h$.

On the other hand, the diagonal elements of $M_n$ are $A_{1,\dots,1}, \dots, A_{n,\dots,n}$, while the off-diagonal elements of $M_n$ do not involve $A_{1,\dots,1}, \dots, A_{n,\dots,n}$. By the properties of determinants, any term of $D_n$ is either the product of all of its diagonal elements, or a product at least missing two diagonal elements, i.e., there does not exist a term of $D_n$, for which the total degree of $A_{1,\dots,1}, \dots, A_{n,\dots,n}$ is $N - 1$. This contradicts the existence of the term (9). This proves the proposition. $\square$

Let $n = 2$. Then we may denote the distinct entries of $A$ as

$$a_0 = A_{1,\dots,1,1},\ a_1 = A_{1,\dots,1,2},\ \dots,\ a_m = A_{2,\dots,2,2}.$$

By Proposition 1 and the Sylvester Formula (Page 400 of Gelfand et al. (1994)), we have the following proposition.

**Proposition 5.** *If $n = 2$, then with the notation above, $\det(A)$ is equal to the following $2(m-1)$-dimensional determinant:*

$$\begin{vmatrix}
a_0 & \binom{m-1}{1}a_1 & \binom{m-1}{2}a_2 & \cdots & \binom{m-1}{m-2}a_{m-2} & a_{m-1} & 0 & 0 & \cdots & 0 \\
0 & a_0 & \binom{m-1}{1}a_1 & \cdots & \binom{m-1}{m-3}a_{m-3} & \binom{m-1}{m-2}a_{m-2} & a_{m-1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_0 & \binom{m-1}{1}a_1 & \binom{m-1}{2}a_2 & \cdot & \binom{m-1}{m-2}a_{m-2} & a_{m-1} \\
a_1 & \binom{m-1}{1}a_2 & \binom{m-1}{2}a_3 & \cdots & \binom{m-1}{m-2}a_{m-1} & a_m & 0 & 0 & \cdots & 0 \\
0 & a_1 & \binom{m-1}{1}a_2 & \cdots & \binom{m-1}{m-3}a_{m-2} & \binom{m-1}{m-2}a_{m-1} & a_m & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & a_1 & \binom{m-1}{1}a_2 & \binom{m-1}{2}a_3 & \cdot & \binom{m-1}{m-2}a_{m-1} & a_m
\end{vmatrix},$$

*where*

$$\binom{m-1}{i} = \frac{(m-1)!}{i!(m-1-i)!}.$$

With some computation, we have the following explicit formula for $\det(A)$ and $\phi(\lambda)$ when $m = 4$ and $n = 2$.

**Corollary 2.** *When $m = 4$ and $n = 2$,*

$$\begin{aligned}
\det(A) = {} & a_0^3a_4^3 - 64a_1^3a_3^3 - 27a_0^2a_3^4 - 27a_1^4a_4^2 - 18a_0^2a_2^2a_4^2 \\
& + 36a_1^2a_2^2a_3^2 + 81a_0a_2^4a_4 - 54a_0a_2^3a_3^2 - 54a_1^2a_2^3a_4 - 12a_0^2a_1a_3a_4^2 \\
& - 6a_0a_1^2a_3^2a_4 + 54a_0^2a_2a_3^2a_4 + 54a_0a_1^2a_2a_4^2 \\
& + 108a_0a_1a_2a_3^3 + 108a_1^3a_2a_3a_4 - 180a_0a_1a_2^2a_3a_4
\end{aligned} \tag{10}$$

*and*

$$\begin{aligned}
\phi(\lambda) = {} & (a_0-\lambda)^3(a_4-\lambda)^3 - 64a_1^3a_3^3 - 27(a_0-\lambda)^2a_3^4 - 27a_1^4(a_4-\lambda)^2 \\
& - 18(a_0-\lambda)^2a_2^2(a_4-\lambda)^2 + 36a_1^2a_2^2a_3^2 + 81(a_0-\lambda)a_2^4(a_4-\lambda) \\
& - 54(a_0-\lambda)a_2^3a_3^2 - 54a_1^2a_2^3(a_4-\lambda) - 12(a_0-\lambda)^2a_1a_3(a_4-\lambda)^2 \\
& - 6(a_0-\lambda)a_1^2a_3^2(a_4-\lambda) + 54(a_0-\lambda)^2a_2a_3^2(a_4-\lambda)
\end{aligned}$$

$$+54(a_0-\lambda)a_1^2a_2(a_4-\lambda)^2+108(a_0-\lambda)a_1a_2a_3^3+108a_1^3a_2a_3(a_4-\lambda)$$
$$-180(a_0-\lambda)a_1a_2^2a_3(a_4-\lambda). \tag{11}$$

## 3. The characteristic polynomial and the E-characteristic polynomial

We may denote the sum of the diagonal elements of $A$ as $\mathrm{tr}(A)$. We call those eigenvalues **N-eigenvalues** of $A$ if they are not H-eigenvalues, i.e., an N-eigenvalue is an eigenvalue which has no real eigenvectors. We call the eigenvectors associated with N-eigenvalues N-eigenvectors.

We now prove the main theorem of this paper.

**Theorem 1.** *We have the following conclusions on eigenvalues of $A$:*

(a) *A number $\lambda \in \mathbf{C}$ is an eigenvalue of $A$ if and only if it is a root of the characteristic polynomial $\phi$.*

(b) *The number of eigenvalues of $A$ is $d = n(m-1)^{n-1}$. Their product is equal to* $\det(A)$.

(c) *If $A$ is diagonal, then $A$ has $n$ H-eigenvalues, which are its diagonal elements, with corresponding unit vectors as their H-eigenvectors. Each of these H-eigenvalues is of multiplicity $(m-1)^{n-1}$, and $A$ has no N-eigenvalues.*

(d) *The sum of all the eigenvalues of $A$ is*

$$(m-1)^{n-1}\mathrm{tr}(A).$$

**Proof.** (a) According to our definition of the symmetric hyperdeterminant, $\phi(\lambda) = 0$ if and only if there is a nonzero vector $x \in \mathbf{C}^n$ such that

$$F(x) \equiv (A-\lambda I)x^m = 0$$

and $\nabla F(x) = 0$. But

$$\nabla F(x) = m(A-\lambda I)x^{m-1} = m\left(Ax^{m-1} - \lambda x^{[m-1]}\right).$$

Then $\nabla F(x) = 0$ is equivalent to (2), while $F(x) = 0$ is equivalent to

$$Ax^m = \lambda \sum_{i=1}^n x_i^m,$$

which is a consequence of (2). The conclusion follows.

(b) By the knowledge of the symmetric hyperdeterminant, the degree of $\phi$ is $d = n(m-1)^{n-1}$. By (4) and Corollary 1, the leading coefficient of $\phi$, i.e., the coefficient of $\lambda^d$, is

$$(-1)^d \det(I) = (-1)^d \neq 0.$$

The first conclusion of (b) follows. The leading coefficient of $\phi$ is $(-1)^d$. The constant term of $\phi$ is $\det(A)$. The second conclusion of (b) then follows from the relation between roots and coefficients of a one-dimensional polynomial.

(c) This follows from (2), (b) of this theorem, and Proposition 3.

(d) By Proposition 4 and the structure of $\phi(\lambda) = \det(A - \lambda I)$, the term of $\lambda^{d-1}$ in $\phi(\lambda)$, where $d = n(m-1)^{n-1}$, is in the term

$$\prod_{i=1}^{n} \left(A_{i,\ldots,i} - \lambda\right)^{(m-1)^{n-1}}.$$

Thus, the coefficient of this term is

$$-(m-1)^{n-1}\mathrm{tr}(A).$$

Since the coefficient of the term $\lambda^d$ in $\phi(\lambda)$ is 1, the conclusion follows from the relation between roots and coefficients of a one-dimensional polynomial. $\square$

**Example 1.** Let $m = 4$ and $n = 2$. Assume that $A_{1111} = A_{2222} = 1$, $A_{1112} = A_{1121} = A_{1211} = A_{2111} = a \neq 0$ and all other $A_{i_1,i_2,i_3,i_4} = 0$. Then from (2), we may directly find that $A$ has four H-eigenvalues and two N-eigenvalues: a double H-eigenvalue $\lambda_1 = \lambda_2 = 1$ with an H-eigenvector $x^{(1)} = (0, 1)^{\mathrm{T}}$, an H-eigenvalue $\lambda_3 = 1 + (27)^{\frac{1}{4}} a$ with an H-eigenvector $x^{(3)} = ((3)^{\frac{1}{4}}, 1)^{\mathrm{T}}$, an H-eigenvalue $\lambda_4 = 1 - (27)^{\frac{1}{4}} a$ with an H-eigenvector $x^{(4)} = ((3)^{\frac{1}{4}}, -1)^{\mathrm{T}}$, an N-eigenvalue $\lambda_5 = 1 + (27)^{\frac{1}{4}} a\sqrt{-1}$ with an N-eigenvector $x^{(5)} = ((3)^{\frac{1}{4}}, \sqrt{-1})^{\mathrm{T}}$, and an N-eigenvalue $\lambda_6 = 1 - (27)^{\frac{1}{4}} a\sqrt{-1}$ with an N-eigenvector $x^{(6)} = ((3)^{\frac{1}{4}}, -\sqrt{-1})^{\mathrm{T}}$. We see that the total number of eigenvalues is

$$d = n(m-1)^{n-1} = 6,$$

the product of all the eigenvalues is $1 - 27a^4$, and the sum of all the eigenvalues is 6. On the other hand, by (10), we have

$$\det(A) = 1 - 27a^4.$$

This is equal to the product of all the eigenvalues. Also

$$(m-1)^{n-1}\mathrm{tr}(A) = 6,$$

which is equal to the sum of all the eigenvalues. By (11), we also have

$$\phi(\lambda) = (1-\lambda)^2[(1-\lambda)^4 - 27a^4],$$

which has six roots $\lambda_i$ for $i = 1, \ldots, 6$ as indicated above.

**Corollary 3.** *Suppose that $B = a(A + bI)$, where $a$ and $b$ are two real numbers. Then $\mu$ is an eigenvalue (H-eigenvalue) of $B$ if and only if $\mu = a(\lambda + b)$ and $\lambda$ is an eigenvalue (H-eigenvalue) of $A$. In this case, they have the same eigenvectors.*

We may use Theorem 1(b) to calculate the symmetric hyperdeterminant. We may also use this property to construct the formulas for the symmetric hyperdeterminant for some sparse tensors. In Section 5, we give an example for this when $m = 4$ and $n = 3$.

We have a conjecture on eigenvalues:

**Conjecture 1.** *The number of linearly independent eigenvectors associated with an eigenvalue $\lambda$ is not greater than the multiplicity of $\lambda$.*

Assume that $m$ is even and let $m = 2l$. As we said in the introduction, in this case, E-eigenvalues are roots of another one-dimensional polynomial associated with $A$, and we call that one-dimensional polynomial the E-characteristic polynomial of $A$. Let $I_2$ be the $n \times n$ unit matrix. To define the E-characteristic polynomial, we need to study a special $m$th-order $n$-dimensional tensor $I_2^l$, whose $(i_1, i_2, \ldots, i_m)$ entry is defined as

$$\delta_{i_1 i_2} \delta_{i_3 i_4} \cdots \delta_{i_{m-1} i_m}.$$

The tensor $I_2^l$ can be regarded as the tensor product of $l$ unit matrices $I_2, \ldots, I_2$. It is not supersymmetric when $l \geq 2$. Let $I_E = \mathrm{sym}(I_2^l)$. We call the one-dimensional polynomial $\psi$, defined by

$$\psi(\lambda) = \det(A - \lambda I_E),$$

the **E-characteristic polynomial** of $A$.

**Proposition 6.** *If $l \geq 2$, then*

$$\det(I_E) = 0.$$

**Proof.** Let $x_1 = 1$, $x_2 = \sqrt{-1}$, $x_i = 0$ for $i = 3, \ldots, n$. Then we see that

$$I_E x^{m-1} \equiv I_2^l x^{m-1} = 0$$

when $l \geq 2$. Hence, 0 is an eigenvalue of $I_2^l$. By Theorem 1(b), we have the conclusion. □

We say that $A$ is **regular** if either $A$ is not singular, or $A$ is singular but there is no eigenvector $x$ associated with the zero eigenvalue of $A$ such that $x \neq 0$ and

$$\sum_{i=1}^{n} x_i^2 = 0.$$

**Theorem 2.** *Assume that $m$ is even and $m = 2l$. We have the following conclusions on E-eigenvalues of $A$:*

(a) *An E-eigenvalue of $A$ is a root of the E-characteristic polynomial $\psi$. If $A$ is regular, then a complex number is an E-eigenvalue of $A$ if and only if it is a root of $\psi$.*

(b) *When $l \geq 2$, the number of E-eigenvalues of $A$ is strictly less than $d = n(m-1)^{n-1}$.*

**Proof.** (a) According to our definition of the symmetric hyperdeterminant, $\psi(\lambda) = 0$ if and only if there is a nonzero vector $x \in \mathbf{C}^n$ such that

$$G(x) \equiv (A - \lambda I_E)x^m \equiv (A - \lambda I_2^l)x^m = Ax^m - \lambda \left( \sum_{i=1}^{n} x_i^2 \right)^l = 0$$

and $\nabla G(x) = 0$. We have

$$\nabla G(x) = m \left( Ax^{m-1} - \lambda x \left( \sum_{i=1}^{n} x_i^2 \right)^{l-1} \right). \tag{12}$$

Then (3) implies that $\nabla G(x) = 0$, while $G(x) = 0$ is equivalent to

$$Ax^m = \lambda \left( \sum_{i=1}^{n} x_i^2 \right)^l,$$

which is also a consequence of (3). The first conclusion follows. Suppose now that $A$ is regular. If

$$\sum_{i=1}^{n} x_i^2 = 0$$

in (12), then $\nabla G(x) = 0$ implies $Ax^{m-1} = 0$, which implies that $A$ is not regular, a contradiction. Hence,

$$\sum_{i=1}^{n} x_i^2 \neq 0$$

in (12). By scaling $x$, we see that $\nabla G(x) = 0$ implies (3) in this case. The second conclusion follows.

(b) By the knowledge of the symmetric hyperdeterminant, the degree of $\psi$ is at most $d = n(m-1)^{n-1}$. But the coefficient of the $d$th-degree term of $\psi$ is $\det(I_E)$, which is zero, according to Proposition 6. Hence, the actual degree of $\psi$ is strictly less than $d$. The conclusion follows. □

We have a conjecture on E-eigenvalues:

**Conjecture 2.** *When $l \geq 2$, the number of E-eigenvalues of $A$ is strictly less than $d = n(m-1)^{n-1} - 1$.*

## 4. More properties of eigenvalues and E-eigenvalues

We have the following theorem.

**Theorem 3.** *Suppose that $x$ is an eigenvector associated with an eigenvalue $\lambda$ of $A$. If*

$$\sum_{i=1}^{n} x_i^m \neq 0, \tag{13}$$

*then*

$$\lambda = \frac{Ax^m}{\sum_{i=1}^{n} x_i^m}. \tag{14}$$

*On the other hand, if $x$ is an E-eigenvector associated with an E-eigenvalue $\lambda$ of $A$, then*

$$\lambda = Ax^m. \tag{15}$$

**Proof.** By (2), we have

$$Ax^m = \lambda \left(x^{[m-1]}\right)^{\mathrm{T}} x = \lambda \left(\sum_{i=1}^{n} x_i^m\right).$$

If (13) holds, then we have (14). We may prove (15) from (3) directly. □

For eigenvectors of two distinct eigenvalues of $A$, we have the following theorem.

**Theorem 4.** *Suppose that $\lambda$ and $\mu$ are two distinct eigenvalues of $A$, $\lambda \neq \mu$, and $x$ and $y$ are two eigenvectors associated with them. Then $x$ and $y$ are linearly independent.*

**Proof.** Suppose that $x$ and $y$ are linearly dependent. Then $x$ and $y$ are proportional, since both of them are nonzero vectors. Since (2) is homogeneous, we see that $x$ is also an eigenvector of $A$ associated with $\mu$. Since $x \neq 0$, there exists $i$ such that $x_i \neq 0$. Consider the $i$th equation of (2); we have

$$\left[Ax^{m-1}\right]_i = \lambda x_i^{m-1} = \mu x_i^{m-1}.$$

Since $x_i \neq 0$, this implies that $\lambda = \mu$, contradicting our assumption. □

**Theorem 5.** *Assume that $m$ is even. The following conclusions hold for $A$:*

(a) *$A$ always has H-eigenvalues. $A$ is positive definite (positive semidefinite) if and only if all of its H-eigenvalues are positive (nonnegative).*

(b) *$A$ always has Z-eigenvalues. $A$ is positive definite (positive semidefinite) if and only if all of its Z-eigenvalues are positive (nonnegative).*

(c) *If $A$ is a multiple of $I$, then $A$ has a $d$-multiple H-eigenvalue, where $d = n(m-1)^{n-1}$. If $A$ is not a multiple of $I$, then $A$ has at least two distinct H-eigenvalues.*

**Proof.** (a) We see that (2) is the optimality condition of

$$\max \left\{ Ax^m : \sum_{i=1}^{n} x_i^m = 1, x \in \mathbf{R}^n \right\} \tag{16}$$

and

$$\min \left\{ Ax^m : \sum_{i=1}^{n} x_i^m = 1, x \in \mathbf{R}^n \right\}. \tag{17}$$

As the feasible set is compact and the objective function is continuous, the global maximizer and minimizer always exist. This shows that (2) always has real solutions, i.e., $A$ always has H-eigenvalues. Since $A$ is positive definite (positive semidefinite) if and only if the optimal value of (17) is positive (nonnegative), we have the second conclusion of (a).

(b) The proof of (b) is similar to the proof of (a), as long as we replace (16) and (17) by

$$\max \left\{ Ax^m : \sum_{i=1}^{n} x_i^2 = 1, x \in \mathbf{R}^n \right\}$$

and

$$\min\left\{Ax^m : \sum_{i=1}^{n} x_i^2 = 1, x \in \mathbf{R}^n\right\}.$$

(c) The first conclusion follows from Theorem 1(c). Suppose that $A$ has only one H-eigenvalue $\lambda$. Since (2) is the optimality condition of (16) and (17), this implies that for any $x \in \mathbf{R}^n$, (2) holds for that $\lambda$. Consider the $i$th equation of (2). Letting $x_j = \delta_{ij}$ for $j = 1, \ldots, n$, we have

$$A_{i,\ldots,i} = \lambda.$$

Only letting $x_i = 1$ and combining with the above, we have

$$\sum_{\substack{i_2,\ldots,i_m=1 \\ \delta_{i,i_2,\ldots,i_m}}}^{n} A_{i,i_2,\ldots,i_m} x_{i_2} \cdots x_{i_m} = 0$$

for any $x$ satisfying $x_i = 1$. This implies that

$$A_{i,i_2,\ldots,i_m} = \lambda \delta_{i,i_2,\ldots,i_m},$$

for any $i, i_2, \ldots, i_m$, i.e., $A$ is a multiple of $I$. This completes the proof. $\square$

**Example 2.** Let $m = 4$ and $n = 2$. Assume that $A_{1111} = 1$, $A_{1122} = A_{1221} = A_{1212} = A_{2121} = A_{2211} = A_{2112} = \frac{1}{3}$, $A_{2222} = 1$ and other $A_{i_1,i_2,i_3,i_4} = 0$. Then (2) becomes

$$\begin{cases} x_1^3 + x_1 x_2^2 = \lambda x_1^3, \\ x_1^2 x_2 + x_2^3 = \lambda x_2^3. \end{cases}$$

Solving it, we find that $A$ has four H-eigenvalues: $\lambda_1 = \lambda_2 = 1$ with H-eigenvectors $x^{(1)} = (1,0)^{\mathrm{T}}$, $x^{(2)} = (0,1)^{\mathrm{T}}$, $\lambda_3 = \lambda_4 = 2$ with H-eigenvectors $x^{(3)} = (1,1)^{\mathrm{T}}$, $x^{(4)} = (1,-1)^{\mathrm{T}}$, and a double zero N-eigenvalue $\lambda_5 = \lambda_6 = 0$ with N-eigenvectors $x^{(5)} = (1, \sqrt{-1})^{\mathrm{T}}$, $x^{(2)} = (1, -\sqrt{-1})^{\mathrm{T}}$. Hence, $A$ is positive definite but singular in the sense of Cayley.

It is seen that if $\lambda$ is an N-eigenvalue of $A$ and $x$ is an eigenvector associated with $\lambda$, then $\bar{\lambda}$ is also an N-eigenvalue of $A$ and $\bar{x}$ is an eigenvector associated with $\bar{\lambda}$. This indicates that N-eigenvalues appear in pairs. Furthermore, the product of a conjugate pair of nonzero N-eigenvalues is always positive. Hence, the sign of $\det(A)$ is the same as the sign of the product of all H-eigenvalues and zero N-eigenvalues if there are any. Thus, we have the following proposition.

**Proposition 7.** *Assume that $m$ is even. If $A$ is positive semidefinite, then* $\det(A) \geq 0$. *If $A$ is positive definite, then either* $\det(A) > 0$ *or $A$ has some zero N-eigenvalues.*

It is easy to see that all of positive semidefinite supersymmetric tensors of the same order and dimension form a closed convex cone. By Proposition 7, the tensors are positive definite in the interior of this convex cone and on some boundary part of this cone.

In (1), if we let some (but not all) $x_i$ be zero, then we have a lower degree homogeneous polynomial, which defines a lower order supersymmetric tensor. We call such a lower order supersymmetric tensor a **principal supersymmetric sub-tensor** of $A$. If $A$ is positive definite (semidefinite), then all of its principal supersymmetric sub-tensors are positive definite (semidefinite). By Proposition 7, we have the following further proposition.

**Proposition 8.** *Assume that $m$ is even. If $A$ is positive semidefinite, then the symmetric hyperdeterminants of all of its principal supersymmetric sub-tensors are nonnegative.*

Note that the converse of Proposition 8 is not true in general. For example, a necessary and sufficient condition for positive definiteness in the case $m = 4$ and $n = 2$ can be found in Jury and Mansour (1981). For positive semidefiniteness, we may take the closed hull form of the condition given in Jury and Mansour (1981), which is much more complicated than the condition in Proposition 8.

## 5. Distribution of the eigenvalues

The following is a theorem on the distribution of the eigenvalues of $A$.

**Theorem 6.** (a) *The eigenvalues of $A$ lie in the union of $n$ disks in* **C**. *These $n$ disks have the diagonal elements of the supersymmetric tensor as their centers, and the sums of the absolute values of the off-diagonal elements as their radii.*

(b) *If one of these $n$ disks is disjoint with the other $n - 1$ disks, then there are exactly $(m - 1)^{n-1}$ eigenvalues which lie in this disk, and when $m$ is even there is at least one H-eigenvalue which lies in this disk.*

(c) *If $k$ of these $n$ disks are connected but disjoint with the other $n - k$ disks, then there are exactly $k(m-1)^{n-1}$ eigenvalues which lie in the union of these $k$ disks. Moreover when $m$ is even at least one H-eigenvalue lies in the real interval intersected by this union on the real axis if one of the following three conditions holds:*

(i) *$k$ is odd;*

(ii) *$k$ is even and the other $n - k$ disks are on the left side of this union;*

(iii) *$k$ is even and the other $n - k$ disks are on the right side of this union.*

**Proof.** (a) Suppose that $\lambda$ is an eigenvalue of $A$ with eigenvector $x$. Assume that

$$|x_i| = \max_{j=1,\dots,n} |x_j|.$$

Consider the $i$th equation of (2). We have

$$(\lambda - a_{i,\dots,i})x_i^{m-1} = \sum_{\substack{i_2,\dots,i_m=1 \\ \delta_{i,i_2,\dots,i_m}=0}}^{n} A_{i,i_2,\dots,i_m} x_{i_2} \cdots x_{i_m}.$$

This implies that

$$|\lambda - a_{i,\dots,i}| \le \sum_{\substack{i_2,\dots,i_m=1 \\ \delta_{i,i_2,\dots,i_m}=0}}^{n} \left|A_{i,i_2,\dots,i_m}\right| \cdot \left|\frac{x_{i_2}}{x_i}\right| \cdots \left|\frac{x_{i_m}}{x_i}\right| \le \sum_{\substack{i_2,\dots,i_m=1 \\ \delta_{i,i_2,\dots,i_m}=0}}^{n} \left|A_{i,i_2,\dots,i_m}\right|.$$

This gives us the desired result. Note that $\lambda$ and $x$ may be non-real here.

(b) This is a special case of (c) with $k = 1$.

(c) Let $D$ be an $m$th-order $n$-dimensional diagonal tensor whose diagonal elements are the same as those of $A$. Let

$$A(\epsilon) = D + \epsilon(A - D),$$

for $\epsilon \in [0, 1]$. Then $A(0) = D$ and $A(1) = A$. Let

$$\phi_\epsilon(\lambda) = \det(A(\epsilon) - \lambda I).$$

Then $\phi_\epsilon$ is a one-dimensional monic polynomial whose coefficients are polynomials of $\epsilon$. Then the roots of $\phi_\epsilon$ are continuous functions of $\epsilon$. Let $\epsilon$ vary from 0 to 1. By (a) of this theorem and Theorem 1(c), we have the first conclusion of (c). (i) If $k$ is odd, then $k(m-1)^{n-1}$ is also odd. When $m$ is even, since N-eigenvalues appear in pairs, there is at least one $H$-eigenvalue in the union. (ii) Consider

$$\max\left\{A(\epsilon)x^m : \sum_{i=1}^n x_i^m = 1,\ x \in \mathbf{R}^n\right\}.$$

Its global minimizers $x(\epsilon)$ are continuous with respect to $\epsilon$. But

$$\lambda(\epsilon) = A(x(\epsilon))^m$$

is an H-eigenvalue. It should stay in the rightmost component of the intersection of the real axis and the union of the $n$ disks. This proves (ii). The proof of (iii) is similar, by changing “min” to “max”. $\square$

When $m$ is even, Theorem 6 gives a lower bound and a new upper bound for the smallest H-eigenvalue, which is useful in judging the positive definiteness of $A$ (Qi, 2004).

**Example 3.** Let $m = 4$ and $n = 3$. Assume that $A_{1111} = 2$, $A_{2222} = 3$, $A_{3333} = 5$, $A_{1123} = A_{1132} = A_{1213} = A_{1312} = A_{1231} = A_{1321} = A_{2113} = A_{3112} = A_{2131} = A_{3121} = A_{2311} = A_{3211} = \frac{a}{3}$ and other $A_{i_1,i_2,i_3,i_4} = 0$.

If $a = 0$, then $A$ is diagonal. By Theorem 1(c), $A$ has three distinct nine-multiple H-eigenvalues, $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 5$, with H-eigenvectors $x^{(1)} = (1, 0, 0)^{\mathrm{T}}$, $x^{(2)} = (0, 1, 0)^{\mathrm{T}}$, $x^{(3)} = (0, 0, 1)^{\mathrm{T}}$, respectively.

Assume that $a \neq 0$. By Theorem 6, the eigenvalues of $A$ lie in the following three disks:

1. Ball 1, with its center at 2 and radius $2|a|$.
2. Ball 2, with its center at 3 and radius $|a|$.
3. Ball 3, with its center at 5 and radius $|a|$.

There are three cases for $a \neq 0$:

(a) $0 < |a| < \frac{1}{3}$. Then $A$ is diagonally dominated (Qi, 2004). In each disk there are 9 eigenvalues, and there are at least one H-eigenvalue in $[2-2|a|, 2+2|a|]$, one H-eigenvalue in $[3 - |a|, 3 + |a|]$ and one H-eigenvalue in $[5 - |a|, 5 + |a|]$. $A$ is positive definite.

(b) $\frac{1}{3} \leq |a| < \frac{2}{3}$. There are 18 eigenvalues in the union of the first two disks, 9 eigenvalues in the third disk. There are at least one H-eigenvalue in $[2 - 2|a|, 3 + |a|]$ and one H-eigenvalue in $[5 - |a|, 5 + |a|]$. $A$ is positive definite.

(c) $|a| \geq \frac{2}{3}$. There are 27 eigenvalues in the union of these three disks. By Theorem 5, there are at least two H-eigenvalues in $[2-2|a|, 5+|a|]$. If $|a| < 1$, then $A$ is positive definite. If $|a| = 1$, then $A$ is positive semidefinite and may be positive definite. If $|a| > 1$, no conclusion can be made on positive definiteness of $A$ by Theorem 6.

We may use (2) to calculate the eigenvalues of $A$. Now (2) becomes

$$\begin{cases} 2x_1^3 + 2ax_1x_2x_3 = \lambda x_1^3, \\ 3x_2^3 + ax_1^2x_3 = \lambda x_2^3, \\ 5x_3^3 + ax_1^2x_2 = \lambda x_3^3. \end{cases} \tag{18}$$

Cancelling $\lambda$ from the first two equations of (18), we have

$$x_1^3x_2^3 = ax_1x_3\left(2x_2^4 - x_1^4\right).$$

Cancelling $\lambda$ from the first and the third equations of (18), we have

$$3x_1^3x_3^3 = ax_1x_2\left(2x_3^4 - x_1^4\right).$$

Cancelling $x_3$ from these two equations, we have

$$3x_1^9x_2^9\left(2x_2^4 - x_1^4\right) = 2x_1^9x_2^{13} - a^4x_1^5x_2\left(2x_2^4 - x_1^4\right)^4. \tag{19}$$

We have a five-multiple root $x_1 = 0$ and a single root $x_2 = 0$.

If $x_2 = 0$ and $x_1 \neq 0$, then $x_3 = 0$ by (18). By (18), we have $\lambda_1 = 2$. We may let $x_1 = 1$.

If $x_1 = 0$ and $x_2 \neq 0$, then $x_3 = 0$ by (18). By (18), we have $\lambda_2 = 3$. We may let $x_2 = 1$.

If $x_1 = 0$ and $x_3 \neq 0$, then $x_2 = 0$ by (18). By (18), we have $\lambda_3 = 5$. We may let $x_3 = 1$.

Thus, when $a \neq 0$, $A$ always has a single H-eigenvalue $\lambda_1 = 2$ with an H-eigenvector $(1, 0, 0)^{\mathrm{T}}$, a five-multiple H-eigenvalue $\lambda_2 = 3$ with an H-eigenvector $(0, 1, 0)^{\mathrm{T}}$, and a five-multiple H-eigenvalue $\lambda_3 = 5$ with an H-eigenvector $(0, 0, 1)^{\mathrm{T}}$.

We now assume that $x_1 \neq 0$ and $x_2 \neq 0$. Let $t = \frac{x_2}{x_1}$ and $s = t^4$. Without loss of generality, we may assume that $x_1 = 1$. Then (19) becomes

$$a^4 = \frac{s^2(3-4s)}{(2s-1)^4}. \tag{20}$$

We have

$$x_2 = t \qquad \text{and} \qquad x_3 = \frac{t^3}{a\left(2t^4 - 1\right)}. \tag{21}$$

By (18) and (21), we have

$$s = \frac{\lambda - 2}{2(\lambda - 3)}. \tag{22}$$

Substituting (22) to (20), we have

$$\eta(\lambda) := (\lambda - 2)^2(\lambda - 3)(\lambda - 5) - 4a^4 = 0.$$

By (21) and $t^4 = s$, we know that each root of $\eta$ is a four-multiple eigenvalue and double H-eigenvalue of $A$. By this and the relations between roots and coefficients of $\eta$, we may conclude that $A$ is positive definite if $|a| < (15)^{\frac{1}{4}}$, positive semidefinite if $|a| = (15)^{\frac{1}{4}}$, not positive semidefinite if $|a| > (15)^{\frac{1}{4}}$, and

$$\det(A) = 2 \times 15^5 \times (60 - 4a^4)^4.$$

It is not difficult to generalize this result to a general case where $A_{1111} = b$, $A_{2222} = c$, $A_{3333} = d$, $A_{1123} = A_{1132} = A_{1213} = A_{1312} = A_{1231} = A_{1321} = A_{2113} = A_{3112} = A_{2131} = A_{3121} = A_{2311} = A_{3211} = \frac{a}{3}$ and other $A_{i_1,i_2,i_3,i_4} = 0$. We may derive a formula for $\det(A)$ as

$$\det(A) = b \times (cd)^5 \times (b^2cd - 4a^4)^4,$$

and a formula for $\phi(\lambda)$ as

$$\phi(\lambda) = (b - \lambda)(c - \lambda)^5(d - \lambda)^5 \left[(b - \lambda)^2(c - \lambda)(d - \lambda) - 4a^4\right]^4.$$

We find that $A$ is positive definite in the interior of the following region:

$$\left\{(a, b, c, d) \in \mathbf{R}^4 : b \geq 0, c \geq 0, d \geq 0, |a| \leq \left(\frac{b^2cd}{4}\right)^{\frac{1}{4}}\right\},$$

positive semidefinite on the boundary of the above region, and not positive semidefinite out of that region.

Furthermore, we see that the sum of all the eigenvalues of $A$ is

$$(m - 1)^{n-1}\mathrm{tr}(A) = 9(b + c + d).$$

From this example, we have four further conjectures on eigenvalues:

**Conjecture 3.** *$A$ has at least $n$ H-eigenvalues.*

**Conjecture 4.** *$A$ has $n$ linearly independent eigenvectors.*

**Conjecture 5.** *$A$ has $n$ linearly independent H-eigenvectors.*

**Conjecture 6.** *If $k$ of the $n$ disks in Theorem 6 are connected but disjoint with the other $n - k$ disks, then there are at least $k$ H-eigenvalues in the interval intersected by the union of these $k$ disks with the real axis.*

Certainly, Conjecture 6 is stronger than Conjecture 3, while Conjecture 5 is stronger than Conjecture 4.

## 6. Orthogonal similarity

Theorem 2(c) and Theorem 6 do not apply to E-eigenvalues and Z-eigenvalues. In particular, a diagonal supersymmetric tensor $A$ may have more than $n$ Z-eigenvalues.

**Example 4.** Let $m = 4$ and $n = 2$. Assume that $A_{1111} = 3$, $A_{1122} = A_{1221} = A_{1212} = A_{2121} = A_{2211} = A_{2112} = a$, $A_{2222} = 1$ and other $A_{i_1,i_2,i_3,i_4} = 0$. Then (3) is

$$\begin{cases} x_2\left(3x_1^3 + 3ax_1x_2^2\right) = x_1\left(3ax_1^2x_2 + x_2^3\right) \\ x_1^2 + x_2^2 = 1. \end{cases} \tag{23}$$

We see that $x_2 = 0$ is its solution. Otherwise, the first equation of (23) gives us

$$3t^3 + 3at = t\left(3at^2 + 1\right),$$

i.e.,

$$t\left((3a-3)t^2 + 1 - 3a\right) = 0, \tag{24}$$

which always has a real root $t_2 = 0$. When $a > 1$ or $a < \frac{1}{3}$, (24) has two more real double roots $t_3 = \sqrt{\frac{3a-1}{3a-3}}$ and $t_4 = -t_3$. Substituting them to

$$x_1^2 + x_2^2 = 1$$

and

$$\lambda = Ax^4 = 3x_1^4 + 6ax_1^2x_2^2 + x_2^4,$$

we find that $A$ always has two Z-eigenvalues: $\lambda_1 = 3$ with a Z-eigenvector $x^{(1)} = e^{(1)} = (1,0)^{\mathrm{T}}$ and $\lambda_2 = 1$ with a Z-eigenvector $x^{(2)} = e^{(2)} = (0,1)^{\mathrm{T}}$. When $a > 1$ or $a < \frac{1}{3}$, $A$ has one more double Z-eigenvalue:

$$\lambda_3 = \frac{3\left(9a^3 - 6a^2 - 3a + 2\right)}{2(3a-2)^2}$$

with Z-eigenvector

$$x^{(3)} = \left(\sqrt{\frac{3a-1}{6a-4}}, \sqrt{\frac{3a-3}{6a-4}}\right)^{\mathrm{T}}$$

and

$$x^{(4)} = \left(\sqrt{\frac{3a-1}{6a-4}}, -\sqrt{\frac{3a-3}{6a-4}}\right)^{\mathrm{T}}.$$

We see that when $a > 1$ or $a < \frac{1}{3}$, $\psi(a) = 9a^3 - 6a^2 - 3a + 2$ has only one real root $-\frac{1}{\sqrt{3}}$. When $a \le -\frac{1}{\sqrt{3}}$, $\lambda_3 \le 0$. This implies that $A$ is not positive definite in that case. When $a > -\frac{1}{\sqrt{3}}$, $A$ is positive definite. Notice that when $a = 0$, $A$ is a diagonal symmetric tensor.

But in that case, beside the two Z-eigenvalues $\lambda_1$ and $\lambda_2$, which are its diagonal elements, $A$ has an additional Z-eigenvalue $\lambda_3 = \frac{3}{4}$, which is the smallest Z-eigenvalue of $A$.

In fact, we have the following proposition.

**Proposition 9.** *Suppose that $A$ is a diagonal supersymmetric tensor with diagonal elements $a_1, \ldots, a_n$. Let*

$$J_1 = \{i : a_i < 0\}, \quad J_2 = \{i : a_i > 0\}.$$

*If at least one of $J_1$ and $J_2$ has more than one element, then $A$ has more than $n$ Z-eigenvalues. In this case, beside the $n$ Z-eigenvalues which are the diagonal elements of $A$, for each $\bar{J}_k \subseteq J_k$ with $|\bar{J}_k| \geq 2$, $k = 1, 2$,*

$$\lambda = (-1)^k \left[ \frac{1}{\sum_{i \in \bar{J}_k} \left( \frac{1}{|a_i|} \right)^{\frac{2}{m-2}}} \right]^{\frac{m-2}{2}}$$

*is a Z-eigenvalue of $A$, with a Z-eigenvector $x$ defined by*

$$x_i = \begin{cases} \left( \frac{\lambda}{a_i} \right)^{\frac{1}{m-2}}, & \text{for } i \in \bar{J}_k, \\ 0, & \text{otherwise.} \end{cases}$$

This proposition may be proved by definitions directly. We omit its proof.

Example 4 and Proposition 9 reveal the dark side of E-eigenvalues (Z-eigenvalues). One may think of giving up E-eigenvalues (Z-eigenvalues). However, in the remaining part of this section, we will show the bright side of E-eigenvalues. This is an orthogonal similarity, which eigenvalues do not have when $m \geq 4$.

Let $P = (p_{ij})$ be an $n \times n$ real matrix. Define $B = P^m A$ as an $m$th-order $n$-dimensional tensor with its entries as

$$B_{i_1,\ldots,i_m} = \sum_{j_1,\ldots,j_m=1}^{n} p_{i_1 j_1} \cdots p_{i_m j_m} A_{j_1,\ldots,j_m}.$$

**Proposition 10.** *$B = P^m A$ defined above is also a supersymmetric tensor.*

**Proof.** Let $\{i_{k_1}, \ldots, i_{k_m}\} = \{i_1, \ldots, i_m\}$. Then

$$\begin{aligned} B_{i_1,\ldots,i_m} &= \sum_{j_1,\ldots,j_m=1}^{n} p_{i_1 j_1} \cdots p_{i_m j_m} A_{j_1,\ldots,j_m} \\ &= \sum_{j_{k_1},\ldots,j_{k_m}=1} p_{i_{k_1} j_{k_1}} \cdots p_{i_{k_m} j_{k_m}} A_{j_{k_1},\ldots,j_{k_m}} = B_{i_{k_1},\ldots,i_{k_m}}. \end{aligned}$$

This proves the proposition. □

**Proposition 11.** *Let $P = (p_{ij})$ be an $n \times n$ real nonsingular matrix. Let $Q = (q_{ij}) = P^{-1}$. If $B = P^m A$, then $A = Q^m B$.*

**Proof.** Let the entries of $C = Q^m B$ be $C_{i_1,\ldots,i_m}$. Then

$$\begin{aligned}
C_{i_1,\ldots,i_m} &= \sum_{j_1,\ldots,j_m=1}^{n} q_{i_1 j_1} \cdots q_{i_m j_m} B_{j_1,\ldots,j_m} \\
&= \sum_{j_1,\ldots,j_m=1}^{n} q_{i_1 j_1} \cdots q_{i_m j_m} \left( \sum_{k_1,\ldots,k_m=1}^{n} p_{j_1 k_1} \cdots p_{j_m k_m} A_{k_1,\ldots,k_m} \right) \\
&= \sum_{k_1,\ldots,k_m=1}^{n} \left( \sum_{j_1=1}^{n} q_{i_1 j_1} p_{j_1 k_1} \right) \cdots \left( \sum_{j_m=1}^{n} q_{i_m j_m} p_{j_m k_m} \right) A_{k_1,\ldots,k_m} \\
&= \sum_{k_1,\ldots,k_m=1}^{n} \delta_{i_1 k_1} \cdots \delta_{i_m k_m} A_{k_1,\ldots,k_m} = A_{i_1,\ldots,i_m}.
\end{aligned}$$

This proves the proposition. □

If $P$ is a real orthogonal matrix and $B = P^m A$, then $A = (P^{\mathrm{T}})^m B$. In this case, we say that $A$ and $B$ are **orthogonally similar**.

**Theorem 7.** *If supersymmetric tensors $A$ and $B$ are orthogonally similar, then they have the same E-eigenvalues. In particular, if $B = P^m A$, $\lambda$ is an E-eigenvalue of $A$ and $x$ is an E-eigenvector of $A$ associated with $\lambda$, where $P$ is an $n \times n$ real orthogonal matrix, then $\lambda$ is also an E-eigenvalue of $B$ and $y = Px$ is an E-eigenvector of $B$ associated with $\lambda$.*

**Proof.** Suppose that $B = P^m A$, $\lambda$ is an E-eigenvalue of $A$ and $x$ is an E-eigenvector of $A$ associated with $\lambda$, where $P = (p_{ij})$ is an $n \times n$ real orthogonal matrix. Let $y = Px$. Then $x = P^{\mathrm{T}} y$, $y^{\mathrm{T}} y = x^{\mathrm{T}} x = 1$ and for $i_1 = 1, \ldots, n$,

$$\sum_{i_2,\ldots,i_m=1}^{n} A_{i_1,\ldots,i_m} x_{i_2} \cdots x_{i_m} = \lambda x_{i_1}.$$

Hence, for $j_1 = 1, \ldots, n$,

$$\begin{aligned}
\lambda y_{j_1} &= \lambda \sum_{i_1=1}^{n} p_{j_1 i_1} x_{i_1} = \sum_{i_1=1}^{n} p_{j_1 i_1} \left( \lambda x_{i_1} \right) \\
&= \sum_{i_1=1}^{n} p_{j_1 i_1} \left( \sum_{i_2,\ldots,i_m=1}^{n} A_{i_1,\ldots,i_m} x_{i_2} \cdots x_{i_m} \right) \\
&= \sum_{i_1=1}^{n} p_{j_1 i_1} \left( \sum_{i_2,\ldots,i_m=1}^{n} A_{i_1,\ldots,i_m} \left( \sum_{j_2=1}^{n} p_{j_2 i_2} y_{j_2} \right) \cdots \left( \sum_{j_m=1}^{n} p_{j_m i_m} y_{j_m} \right) \right) \\
&= \sum_{j_2,\ldots,j_m=1}^{n} \left( \sum_{i_1,\ldots,i_m=1}^{n} p_{j_1 i_1} \cdots p_{j_m i_m} A_{i_1,\ldots,i_m} \right) y_{j_2} \cdots y_{j_m} \\
&= \sum_{j_2,\ldots,j_m=1}^{n} B_{j_1,\ldots,j_m} y_{j_2} \cdots y_{j_m}.
\end{aligned}$$

This shows that $\lambda$ is also an E-eigenvalue of $B$ and $y = Px$ is an E-eigenvector of $B$ associated with $\lambda$. This proves the theorem. □

## 7. Concluding remarks

In this paper, we defined the symmetric hyperdeterminant, eigenvalues and eigenvectors of a real supersymmetric tensor $A$, and discussed their properties. We see that they have a clear harmonic structure, with a close link with the positive definiteness issue. The more we know about them, the more capable we are of solving the positive definiteness issue. We have also made six conjectures for further exploration.

Assume that $m$ is even. When $n$ and $m$ are small, by the theory of resultants (Cox et al., 1998; D'Andrea and Dickenstein, 2001; Gelfand et al., 1994; Sturmfels, 2002) or the theory of bracket algebra (Cox et al., 1998; Sturmfels, 1993), it is possible to have the formula for the symmetric hyperdeterminant, and hence to find the characteristic polynomial $\phi$. We may find the smallest real root of $\phi$. If it is positive, then $A$ is positive definite. If it is not positive and is of odd multiplicity, then $A$ is not positive definite. Otherwise, we may try to find whether $\phi$ has a nonpositive root of odd multiplicity. If there is such a root, then $A$ is not positive definite. If $A$ has no nonpositive roots of odd multiplicity, but has some nonpositive roots of even multiplicity, then we need to identify whether these roots are H-eigenvalues or N-eigenvalues of $A$, in order to find whether $A$ is positive definite or not. This gives an approach for the positive definiteness issue but further exploration of this aspect is also needed.

Actually, we should not confine the applications of eigenvalues and E-eigenvalues to the positive definiteness issue. For the positive definiteness issue, only the smallest H-eigenvalue and the smallest Z-eigenvalue are important. If we consider the classification and properties of higher order curves (for $n = 2$) and surfaces (for $n = 3$) defined by

$$f(x) \equiv Ax^m = 1,$$

then the other H-eigenvalues and Z-eigenvalues may also play roles. Also, because of the orthogonal similarity, Z-eigenvalues may play a more important role here. This will be our further research topic.

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