Path Sampling: A Fast and Provable Method for Estimating 4-Vertex Subgraph Counts[*]

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ABSTRACT

Counting the frequency of small subgraphs is a fundamental technique in network analysis across various domains, most notably in bioinformatics and social networks. The special case of triangle counting has received much attention. Getting results for 4-vertex patterns is highly challenging, and there are few practical results known that can scale to massive sizes. Indeed, even a highly tuned enumeration code takes more than a day on a graph with millions of edges. Most previous work that runs for truly massive graphs employ clusters and massive parallelization.

We provide a sampling algorithm that provably and accurately approximates the frequencies of all 4-vertex pattern subgraphs. Our algorithm is based on a novel technique of *3-path sampling* and a special pruning scheme to decrease the variance in estimates. We provide theoretical proofs for the accuracy of our algorithm, and give formal bounds for the error and confidence of our estimates. We perform a detailed empirical study and show that our algorithm provides estimates within 1% relative error for all subpatterns (over a large class of test graphs), while being orders of magnitude faster than enumeration and other sampling based algorithms. Our algorithm takes less than a minute (on a single commodity machine) to process an Orkut social network with 300 million edges.

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General Terms

Algorithms, Theory

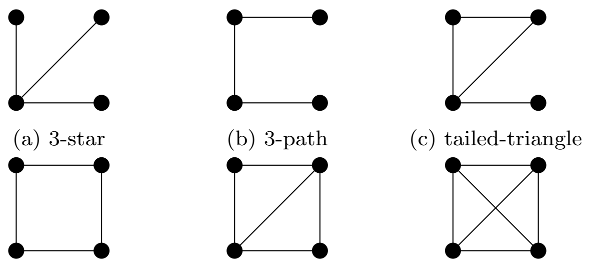
Keywords

Subgraph counting; sampling algorithms; path sampling; motif counting; graphlet distribution

1. INTRODUCTION

Counting the number of occurrences of small subgraphs in a graph is a fundamental network analysis technique used across diverse domains: bioinformatics, social sciences, and infrastructure networks studies [18, 10, 25, 23, 7, 26, 13, 19, 2, 14, 33, 40, 32]. The subgraphs whose counts are desired are variously referred as "pattern subgraphs," "motifs," or "graphlets." It is repeatedly observed that certain small subgraphs occur substantially more often in real-world networks than in a randomly generated network [18, 39, 23]. Motifs distributions have been used in bioinformatics to evaluate network models [26, 19]. Analysis of triadic (3-vertex) motifs has a long history in social network analysis and modeling [18, 7, 13, 30, 12]. Work in the data mining community has applied motif frequencies for spam detection and group classification of sets of nodes [2, 37].

The main challenge of motif counting is combinatorial explosion. Even in a moderately sized graph with millions of edges, the subgraph counts (even for 4-vertex patterns) is in the billions. Any exhaustive enumeration method (no matter how cleverly designed) is forced to touch each occurrence of the subgraph, and cannot truly scale. One may apply massive parallelism to counteract this problem, but that does not avoid the fundamental combinatorial explosion. An alternative approach is based on *sampling.* Here, we try to count the number of subgraphs using a randomized algorithm. The difficulty is in designing a fast algorithm that also provides an accurate estimate. The holy grail is to get mathematically provable bounds on accuracy with quantifiable error bars.


(a) 3-star (b) 3-path (c) tailed-triangle

(d) 4-cycle (e) chordal-4-cycle (f) 4-clique

Figure 1: List of all connected 4-vertex motifs

Sampling approaches have been employed for triangle counting with good success [28, 34, 36, 35, 31]. There also exists work for counting larger motifs, as we shall discuss later. Most methods (especially in bioinformatics) [19, 4, 41, 27] work for graphs of at most 100K edges, much smaller than the massive social networks we encounter.

1.1 The main problem

We focus on estimating frequency of all connected 4-vertex subgraphs on massive input graphs. There are six connected 4-vertex graphs (Fig. 1): (i) the 3-path, (ii) the 3-star, (iii) the tailed-triangle, (iv) the 4-cycle, (v) the chordal-4-cycle, and (vi) the 4-clique. Throughout this work, we refer to these motifs by their numbering in this list. For example the "6-th motif" is the 4-clique.

Our aim is to give an accurate and fast estimate of all 4-vertex subgraph counts. Triadic analysis is now a standard aspect of network analysis. Recent work of Ugander et al [37] specifically use 4-vertex pattern counts to provide a "map" of egonets, and show significant patterns among these counts. Such analyses require fairly precise frequency counts.

1.2 Related Work

Motif counting for bioinformatics was arguably initiated by a seminal paper of Milo *et al.* [23]. This technique has been used for graph modeling [26, 19], graph comparisons [26, 16], and even decomposing a network [20]. Refer to [4, 41] for more details.

Triangle counting has a rich history in social sciences and related analyses, that we simply refer the reader to the related work sections of [35, 31]. The significance of 4-vertex patterns was studied in recent work of Ugander et al. [37], who propose a "coordinate system" for graphs based on the motifs distribution. This is used for improved network classification, and the input graphs were comparatively small (thousands of vertices).

Previous studies tailored to 4-vertex patterns [15, 22] provide both exact and approximation algorithms. However, the asymptotic bounds in these graphs are far from practical, and they are only applied to small graphs. For example, a graph with 90K edges requires 40 minutes of processing [22]. Color coding [1], Monte-Carlo Markov Chain sampling [5], and edge sampling to speedup edge iteration based algorithms [27] have been adopted to count patterns in graphs. We will provide 'detailed comparisons with these methods in §6.1. To exploit more powerful computing platforms, incremental pattern building algorithms for Map-Reduce have been described in [24, 6].

Most relevant to this work are previous studies on *wedge sampling* [28, 31, 21]. This method samples paths of length 2 to estimate various triangle statistics. Our method of 3-path sampling can be seen as building on wedge sampling. We employ new path pruning techniques to improve the algorithm's efficiency. These pruning techniques are inspired by degeneracy ordering algorithms for triangle counting [9, 29]. We can actually provide mathematical error bars for real runs and instances (as opposed to just a theoretical proof of convergence of estimate).

1.3 Summary of our contributions

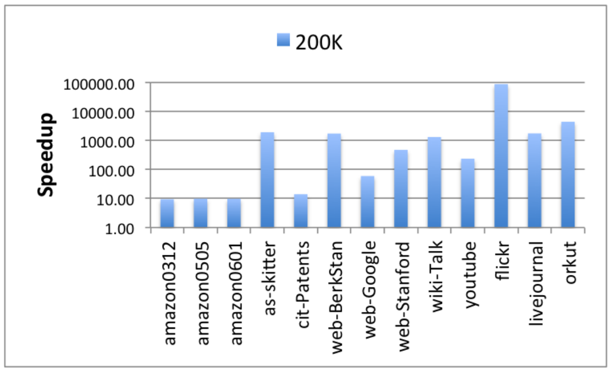
We design a new randomized algorithm, based on *3-path sampling*, that outputs accurate estimates of all 4-vertex subgraphs counts. The algorithm is provably correct and makes no distributional assumption on the graph. All probabilities are over the internal randomness of the algorithm itself (which is independent of the instance). We run detailed simulations on a large variety of datasets, including product co-purchasing networks, web networks, autonomous systems networks, and social networks. All experiments are done on a single commodity machine using 64GB memory.

Extremely fast. Our algorithm relies on a sampling based approach making it extremely fast even on very large graphs. Indeed, there are instances where a finely tuned enumeration code takes almost a day to compute counts of 4-vertex motifs whereas our algorithm only takes less than a minute to output accurate estimates. Refer to Fig. 2a for speedup over a well-tuned enumeration code. Our algorithm takes less a minute on an Orkut social network with 200 million edges, where the total count of each motif is over a billion (and most counts are over 10 billion). An input Flickr social network has more than 10 billion 4-cliques; we get estimate of this number with less than 0.5% error within 30 seconds on a commodity machine. We do not preprocess any of the graphs, and simply read them as a list of edges.

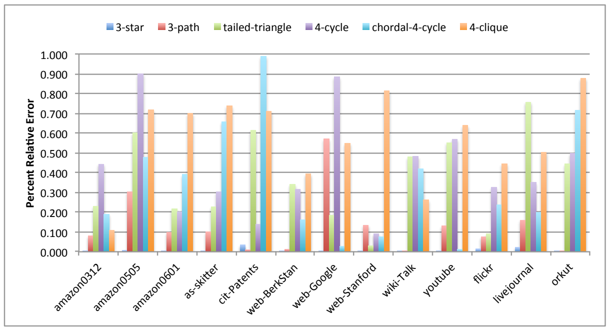
Excellent empirical accuracy. We empirically validate our algorithm on a large variety of datasets, and it consistently gives extremely accurate answers. Refer to Fig. 2b. We get $< 1\%$ relative error *for all subgraph counts on all datasets*, even those with more than 100M edges. (Exact counts were obtained by brute-force enumerations that took several days.) This is much more accurate than any existing method to count such motifs. We compare with existing sampling methods, and demonstrate that our algorithm is faster and more accurate than the state-of-the-art.

Provable guarantees with error bars. Our algorithm has a provable guarantee on accuracy and running time. Furthermore, we can quantify the accuracy/confidence on real inputs and runs of our algorithm. For a given number of samples, we can have a method to put an explicit error bar on our estimate, based on asymptotically tight versions of Chernoff's bound. While these error bars are not as tight as the real errors in Fig. 2b, we can still mathematically prove that the errors are mostly within 5% and always within 10%.

Trends in 4-vertex pattern counts: Given the rapid reporting of 4-vertex pattern counts, our algorithm can be used as a tool for motif analysis. We detect common trends among a large variety of graphs. Not surprisingly, the 3-star is the most frequent 4-vertex motif in all graphs we experimented upon. The least frequent is either the 4-cycle or the 4-clique. The chordal-4-cycle frequency is always more than that of the 4-cycle or 4-clique. Ugander et al [37] study



(a) Speedup



(b) Relative error

Figure 2: Summary of 3-path sampling algorithm behavior over a large variety of datasets: The left figure shows speedup over a tuned enumeration code. The right figure shows the relative error of each estimate, which is always less than 1% (and mostly much smaller).

what trends are merely implied by graph theory, and what are actually features of real-world graphs. Such analyses require accurate estimates quickly, which our algorithm can provide. It is a promising direction to use our algorithm to provide more input to such studies.

2. FORMAL DESCRIPTION OF THE PROBLEM

Our input is an undirected simple graph $G = (V, E)$, with n vertices and m edges. For vertex v, d_v is the degree of v.

It is critical to distinguish subgraphs from *induced subgraphs*. A subgraph is simply some subset of edges. An induced subgraph is obtained by taking a subset V' of vertices, and consider *all edges* among these vertices. Refer to Fig. 3. The edges $(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)$ form a 4-cycle, but the vertex set $\{v_1, v_2, v_3, v_4\}$ induces a chordal-4-cycle. We collectively refer to the 4-vertex subgraphs as "motifs".

It is technically convenient to think of induced subgraph counts. We denote the number of induced occurrences of the i-th subgraph (of Fig. 1) by C_i. So, C_4 is the number of induced 4-cycles in G, which is the number of distinct subsets of 4 vertices that induce a 4-cycle. When we talk of a "vanilla" subgraph, we mean the usual subgraph setting (a subset of edges). In general, if we do not say "induced", we mean vanilla.

Our aim is to get an estimate of all C_i values. Let N_i denote the number of (vanilla) subgraph occurrences of the ith subgraph, There is a simple linear relationship between induced and non-induced counts, given below. The (i, j) entry of the matrix A below is simply the number of distinct copies of the ith subgraph in the jth subgraph (so $A_{2,4} = 4$, the number of 3-paths in the 4-cycle).

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 2 & 4 \\ 0 & 1 & 2 & 4 & 6 & 12 \\ 0 & 0 & 1 & 0 & 4 & 12 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{pmatrix}. \qquad (1)$$

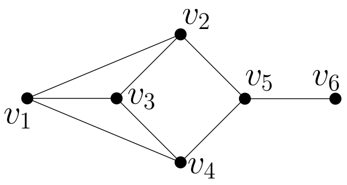


Figure 3: An example graph.

3. THE BASIC ALGORITHM: ESTIMATING COUNTS VIA 3-PATH SAMPLING

Our algorithm for estimating counts of 4-vertex motifs is based on 3-path sampling. In this section, we discuss a basic version of this method. In the next section, we enhance it to improve accuracy.

We begin with a simple procedure that samples a uniform (vanilla) random 3-path. For each edge $e = (u, v) \in E$, denote $\tau_e = (d_u - 1)(d_v - 1)$. We denote $W = \sum_e \tau_e$.

Algorithm 1: `sample`

```
1 Compute $\tau_e$ for all edges and set $p_e = \tau_e/W$.
2 Pick edge $e = (u, v)$ with probability $p_e$.
3 Pick uniform random neighbor $u'$ of $u$ other than $v$.
4 Pick uniform random neighbor $v'$ of $v$ other than $u$.
5 Output the three edges $\{(u', u), (u, v), (v, v')\}$.
```

Observe that the output of `sample` can either be a triangle (if $u' = v'$) or a 3-path.

Claim 3.1. *Fix any 3-path. The probability that* `sample` *outputs this 3-path is exactly* $1/W$.

Proof. Fix a 3-path $(u', u), (u, v), (v, v')$ $(u, u', v, v'$ are all distinct). The probability that $e = (u, v)$ is selected as the middle edge (in Step 2) is exactly $(d_u - 1)(d_v - 1)/W$. Conditioned on this event, the probability that u' is selected as a neighbor of u is $1/(d_u - 1)$ (note that the neighbor v is excluded). Similarly, v' is selected with probability $1/(d_v - 1)$. Putting it all together, the 3-path is chosen with probability $[(d_u - 1)(d_v - 1)/W] \cdot [1/(d_u - 1)] \cdot [1/(d_v - 1)] = 1/W$. The probability is the same for all 3-paths, proving our claim. □

All motifs of Fig. 1, except the 3-star, contain a 3-path. So one can perform the following experiment. Run `sample` to get a collection of edges, and hence a set of (at most 4) vertices. Check the motif induced by this set of vertices. Repeat this experiment many times to estimate the true counts C_i ($i \in [2,6]$). Finally, we use the formula of (1) to estimate C_1. This is exactly the algorithm **`3-path-sampler`**, as given in Alg. 2. We remind the reader that $A_{2,i}$ is the number of 3-paths in the ith motif.

Algorithm 2: `3-path-sampler`
Input: graph $G = (V, E)$, samples k

```
1 Run sample $k$ times to get $k$ sets of edges. Let $S_\ell$
  denote the set of corresponding vertices for the $\ell$th set.
2 Initialize $count_i = 0$ for $i \in [2,6]$.
3 For $\ell \in [1,k]$,
4     Determine subgraph induced by $S_\ell$.
5     If this is the $i$th motif, increment $count_i$.
6 For each $i \in [2,6]$,
7     Set $\widehat{C}_i = (count_i/k) \cdot (W/A_{2,i})$.
8 Set $N_1 = \sum_{v \in V} \binom{d_v}{3}$.
9 Set (induced 3-stars) $\widehat{C}_1 = N_1 - \widehat{C}_3 - 2\widehat{C}_5 - 4\widehat{C}_6$.
```

We prove that **`3-path-sampler`** outputs unbiased estimates for all C_is.

THEOREM 3.2. *For every $i \in [1,6]$, $\mathbf{E}[\widehat{C}_i] = C_i$.*

PROOF. First, let use deal with subgraphs other than the 3-star, so fix some $i \neq 1$. For each $\ell \in [k]$, let X_ℓ be the indicator random variable for S_ℓ inducing the ith motif. So X_ℓ is 1 iff the ℓth call to `sample` outputs a 3-path contained in a copy of the ith motif. The total number of (distinct) such 3-paths is exactly $A_{2,i} \cdot C_i$. By Claim 3.1, the probability that $X_\ell = 1$ is $A_{2,i} \cdot C_i/W$. Hence, $\mathbf{E}[X_\ell] = C_i \cdot A_{2,i}/W$.

We have $\mathbf{E}[count_i] = \sum_{\ell=1}^{k} \mathbf{E}[X_\ell] = (kC_iA_{2,i})/W$, by linearity of expectation. Hence, $\mathbf{E}[\widehat{C}_i] = C_i$. Now, we detail with $\widehat{C}_1$. Note that N_1, the number of 3-star subgraphs, is exactly $\sum_{v \in V} \binom{d_v}{3}$. By linearity of expectation, $\mathbf{E}[\widehat{C}_1] = N_1 - \mathbf{E}[\widehat{C}_3] - 2\mathbf{E}[\widehat{C}_5] - 4\mathbf{E}[\widehat{C}_6]$, which is $N_1 - C_3 - 2C_5 - 4C_6 = C_1$ (as given by (1)). $\square$

We can also prove concentration results using the Hoeffding bound [17]. This is useful as a proof of concept, but does not give useful bounds in practice. (We give more details later.) This analysis is analogous to that of wedge sampling results [29, 31].

THEOREM 3.3 (HOEFFDING [17]). *Let $X_1, X_2, \ldots, X_k$ be independent random variables with $0 \leq X_i \leq 1$ for all $i = 1, \ldots, k$. Define $\bar{X} = \frac{1}{k}\sum_{i=1}^{k} X_i$. Let $\mu = \mathbf{E}[\bar{X}]$. Then for $\varepsilon \in (0,1)$, we have*

$$\Pr[|\bar{X} - \mu| \geq \varepsilon] \leq 2\exp(-2k\varepsilon^2).$$

We can derive concentration results quite directly from this bound. Note that the bound is for a fixed $i \in [2,6]$, i.e., a fixed motif that is not the 3-star.

THEOREM 3.4. *Fix $\delta, \varepsilon \in (0,1)$ and $i \in [2,6]$. Set $k = \lceil (2\varepsilon)^{-2} \ln(2/\delta) \rceil$. For all $i \in [2,6]$: with probability at least $1 - \delta$, $|\widehat{C}_i - C_i| < \varepsilon W/A_{2,i}$. With probability at least $1 - \delta$, $|\widehat{C}_1 - C_1| < \varepsilon W$.*

PROOF. We have fixed $i \neq 1$. For each $\ell \in [k]$, define X_ℓ to be the indicator random variable for S_ℓ inducing the ith motif. Observe that *when i is fixed*, each X_ℓ is independent, since it is simply the result on an independent sample. In other words, the chance of the ℓth sample inducing the ith motif is independent over the sample index ℓ.

Apply Theorem 3.3 to $X_1, \ldots, X_k$. With probability at most δ, $|\bar{X} - \mathbf{E}[\bar{X}]| \geq \varepsilon$ (we use the notation from Theorem 3.3). It remains to interpret $\bar{X}$. Note that $\widehat{C}_i = (count_i/k) \cdot (W/A_{2,i})$. Since $count_i = \sum_{\ell=1}^{k} X_\ell$, $\widehat{C}_i = \bar{X} \cdot (W/A_{2,i})$. So $|\bar{X} - \mathbf{E}[\bar{X}]| \geq \varepsilon$ implies $|\widehat{C}_i - C_i| \geq \varepsilon$, as desired.

Since $\widehat{C}_1$ is obtained by subtracting out other terms, it appears that the errors could add up. With a little care, we can get the same bound as the other $\widehat{C}_i$'s. Define random variable Y_ℓ as follows: if S_ℓ induces a tailed triangle, $Y_\ell = 1/A_{2,3} = 1/2$. If S_ℓ induces a chordal-4-cyle, $Y_\ell = 2/A_{2,5} = 2/6$. If S_ℓ induces a 4-clique, $Y_\ell = 4/A_{2,6} = 4/12$. In all other cases, $Y_\ell = 0$. We have constructed this random variable, so that $\mathbf{E}[Y_\ell] = (C_3 + 2C_5 + 4C_6)/W$.

Observe that $\widehat{C}_1$ can also be expressed as $N_1 - (\sum_\ell Y_\ell/k)W$. The additive error $|\widehat{C}_1 - C_1|$ is the same as $W \cdot |\bar{Y} - \mathbf{E}[\bar{Y}]|$. Applying Theorem 3.3, with probability at least $1 - \delta$, $|\widehat{C}_1 - C_1| < \varepsilon W$. $\square$

To get error bounds for obtaining all estimates, we take the union bound. Hence, with probability at least $1 - 6\delta$, we get the same accuracy guarantee of Theorem 3.4 simultaneously for all counts.

THEOREM 3.5. *The running time (including all preprocessing) of* **`3-path-sampler`** *is $O(m + k \log k)$. The total storage is $O(m + k)$.*

PROOF. We store the graph as an adjacency list. We assume that each list is a hash table, so we can access the degree d_v, find a random neighbor of v, and check for existence of edge (u, v) in (expected over hash table) constant time.

The preprocessing required to determine each value of τ_e is linear in m. Generating k 3-path samples can be done by sorting k random numbers in $(0,1)$ and merging with a list of partial sums of τ_e. This takes $O(k \log k)$ time. (It is possible to get these samples in $O(k)$ using Walker's alias method [38], but it rarely gives a benefit in practice.) Checking the motif induced by the path takes $O(1)$ time by the hash table representation. The total running time is bounded by $O(m + k \log k)$. The only additional storage over the graph are the values τ_e and the various counts. Hence, the storage can be bounded by $O(m + k)$. $\square$

3.1 The challenge of cycle-based motifs

Theorem 3.2 and Theorem 3.4 seem to give us all we want, so why aren't we done? The catch is that the concentration bound of Theorem 3.4 is actually too weak to give reasonable estimates for real world graphs. Let us do some rough calculations, ignoring the constants. To get an estimate such that $|\widehat{C}_i - C_i| < \varepsilon W$, we require $k \approx 1/\varepsilon^2$. But for such an estimate to be useful, we need to understand how W relates to C_i. So ε needs to be of the order of C_i/W, and consequently, k needs to be $(W/C_i)^2$.

Refer to Tab. 1 for the values of W and a few C_is. (For convenience, we just give the order of magnitude of each

Table 1: W vs C_i: counts given as orders of magnitude.

Graph	W	C_2	C_3	C_4	C_6
amazon0312	E+09	E+08	E+08	E+06	E+06
as-skitter	E+12	E+11	E+11	E+10	E+08
orkut	E+13	E+13	E+12	E+10	E+09

number. Full numbers are given later.) For $i \in \{1,2,3\}$ (3-star, 3-path, and tailed triangle), $(W/C_i)^2$ is usually $< 10^4$. This is fairly reasonable number of samples to take, and leads to an efficient and accurate algorithm. On the other hand, for $i \in \{4,5,6\}$ (4-cycle, chordal-4-cycle, and 4-clique), $(W/C_i)^2$ is often $> 10^8$, which is too many samples to take.

In other words, `3-path-sampler` does not perform well for motifs containing a 4-cycle. This leads us to a new algorithm for dealing with these motifs, as described in the next section.

4. IMPROVED ESTIMATION OF 4-CYCLE-BASED MOTIFS VIA *CENTERED* 3-PATHS

We denote the 4-cycle, chordal-4-cycle, and 4-clique as *cycle-based* motifs. We design a better algorithm to estimate them. While the algorithm is provably correct for any graph, the fact that it gives a significant improvement is dependent on the structure of real-world graphs.

Our aim is to find a subset $\mathcal{S}$ of 3-paths with the following properties:

- Every cycle-based motif is guaranteed to contain a fixed number of 3-paths from $\mathcal{S}$.
- It is possible to quickly generate uniform random samples from $\mathcal{S}$.
- $|\mathcal{S}|$ is significantly smaller than $W = \sum_{e=(u,v)}(d_u - 1)(d_v - 1)$.

Let us go back to `sample`, and think of enumerating all 3-paths. For edge (u,v), we take every neighbor of u and every neighbor of v to generate a 3-path. We basically take the Cartesian product of the adjacency lists of u and v. Could we prune the adjacency lists so this product is smaller?

Suppose we order all vertices based on degree and vertex id. So we say $u \prec v$ if: $d_u < d_v$ or, if $d_u = d_v$, the vertex id of u is less than that of v. We could prune the lists using this ordering. When looking for 3-paths where (u,v) is the middle edge, we only look at the portion of u's list "greater" than v, and the portion of v's list greater than u. In general, many 3-paths are generated when d_u and d_v are large. But in that case, we hope that many neighbors of u and v are of lower degree. The pruning ignores such vertices and (hopefully) reduces the set of 3-paths considered. Let us define the set $\mathcal{S}$ of centered 3-paths.

DEFINITION 4.1 (CENTERED 3-PATH). *A 3-path formed by edges $\{(t,u),(u,v),(v,w)\}$ is* centered *if: $v \prec t$, $u \prec w$, and the edge (t,w) exists in the graph (so t,u,v,w form a 4-cycle).*

We prove the important property that every cycle-based motif contains a fixed number of centered 3-paths.

LEMMA 4.2. *Every induced 4-cycle and chordal-4-cycle contains exactly one centered 3-path. Every induced 4-clique contains exactly three centered 3-paths.*

PROOF. Consider any (vanilla) 4-cycle, formed by vertices (in order) t,u,v,w. Pick the smallest vertex, say u. Pick the neighbor of u that is smaller, say v. We show that the 3-path $\{(t,u),(u,v),(v,w)\}$ is the only centered 3-path in this 4-cycle.

By the choice of (u,v), $v \prec t$ and $u \prec w$. Hence, we see that $\{(t,u),(u,v),(v,w)\}$ is centered. The only other possible centered 3-path is $\{(u,t),(t,w),(w,v)\}$. Because $v \prec t$, this path cannot be centered. That completes the proof for the induced 4-cycle case.

Now, suppose t,u,v,w forms an induced chordal-4-cycle. The extra 3-paths contain the chord in the middle, and such 3-paths do not lie on a 4-cycle. So there only exists one centered 3-path.

A 4-clique contains three 4-cycles that partition the 12 different 3-paths. Each of these 4-cycles has a centered 3-path, yielding a total of three such 3-paths. □

We now show how to sample a uniform random centered 3-path. It is quite analogous to `sample`. First, some notation. Let $L_{u,v}$ be the number of neighbors of u greater than v. By sorting all adjacency lists according to vertex degree and id, we can compute for every edge $e = (u,v)$, the value $\lambda_e = L_{u,v}L_{v,u}$. Let $\Lambda = \sum_e \lambda_e$.

Algorithm 3: `sample-centered`

```
1 Compute $\lambda_e$ for all edges and set $p_e = \lambda_e/\Lambda$.
2 Pick edge $e = (u,v)$ with probability $p_e$.
3 Pick uniform random neighbor $u'$ of $u$ such that $v \prec u'$.
4 Pick uniform random neighbor $v'$ of $v$ such that $u \prec v'$.
5 Output the three edges $\{(u',u),(u,v),(v,v')\}$.
```

Note that it is possible that `sample-centered` outputs a 3-path that is not centered (if the 3-path does not lie on a 4-cycle). Nonetheless, analogous to Claim 3.1, we have the following.

CLAIM 4.3. *Fix any centered 3-path. The probability that* `sample-centered` *outputs this 3-path is exactly $1/\Lambda$.*

Now, we give the algorithm that estimates the number of cycle-based motifs. It is analogous to `3-path-sampler`, only using centered 3-paths. For convenience, let B_i denote the number of centered 3-paths in the ith motif. So $B_4 = B_5 = 1$ and $B_6 = 3$, by Lemma 4.2.

Algorithm 4: `centered-sampler`
Input: graph $G = (V,E)$, samples k

```
1 Run sample-centered $k$ times to get $k$ set of edges. Let
  $T_\ell$ denote the set of corresponding edges for the $\ell$th set.
2 Initialize $count_i = 0$ for $i \in [4,6]$.
3 For $\ell \in [1,k]$,
4     If $T_\ell$ is a centered 3-path,
5         Determine subgraph induced by $S_\ell$.
6         If this is the $i$th motif, increment $count_i$.
7 For each $i \in [4,6]$,
8     Set $\widehat{C}_i = (count_i/k) \cdot (\Lambda/B_i)$.
```

Analogous to Theorem 3.4, we can prove the following. Observe how W is replaced by Λ.

THEOREM 4.4. *Fix $\delta, \varepsilon \in (0,1)$ and set $k = \lceil (2\varepsilon)^{-2} \ln(2/\delta) \rceil$. For all $i \in [4,6]$: with probability at least $1-\delta$, $|\widehat{C}_i - C_i| < \varepsilon\Lambda/B_i$.*

For the same number of samples, the performance of `centered-sampler` requires an additional logarithmic factor because of additional preprocessing. In general, d_v is much smaller than n (and is effectively constant for most vertices), so the additional logarithmic factor is not too expense. We require fewer samples for the same accuracy, so `centered-sampler` wins at scale.

THEOREM 4.5. *The running time of* `centered-sampler` *is $O(\sum_v d_v \log d_v + k \log k)$ and the total storage is $O(m+k)$.*

PROOF. As discussed earlier, we need to sort the adjacency lists to determine each value of λ_e. That leads the running time bound. We store adjacency lists as (variable sized) array. This allows us to sort a list of size d_v in time $O(d_v \log d_v)$. The total running time of this step is $O(\sum_v d_v \log d_v)$. Once the lists are sorted, the remaining analysis is identical to that of Theorem 3.5 for `3-path-sampler`. □

4.1 Why centered 3-paths help

We put the value of W and Λ for various real world networks in Tab. 2. Observe how Λ is at least an order of magnitude smaller than W. This is a huge difference when it comes to the sampling bounds in Theorem 3.4 and Theorem 4.4. These bounds show that *two orders of magnitude less samples* suffice for the same error (in estimating cycle-based motifs). This improvement is extremely significant for getting good accuracy with fewer samples.

Table 2: Difference between the number of 3-paths and the number of centered 3-paths.

Graph	W	Λ	W/Λ
amazon0312	1.40E+09	9.36E+07	15
amazon0505	1.59E+09	1.02E+08	16
amazon0601	1.57E+09	1.01E+08	15
as-skitter	1.43E+12	9.05E+10	16
cit-Patents	9.16E+09	8.78E+08	10
web-BerkStan	1.69E+12	1.28E+11	13
web-Google	2.05E+10	6.34E+08	32
web-Stanford	1.85E+11	1.36E+10	14
wiki-Talk	1.31E+12	9.08E+09	144
youtube	1.19E+11	1.68E+09	71
flickr	1.31E+13	8.42E+11	16
livejournal	1.67E+12	1.14E+11	15
orkut	2.22E+13	9.48E+11	23

The final algorithm is simply obtained by running both `3-path-sampler` and `centered-sampler`. The former gives estimates for C_1, C_2, C_3 (we simply discard the remaining estimates), and the latter estimates C_4, C_5, C_6.

5. GETTING PRACTICAL ERROR BARS

While the Hoeffding bound used above provides theoretical convergence, we do not get practical error bars from it. In this section, we show how to get useful error bars for our algorithm on real instances.

All of our sampling algorithms have the same underlying primitive: try to estimate the expectation p of a Bernoulli random variable. We generate a binomial random variable $X \sim B(k,p)$ (by performing k i.i.d. Bernoulli trials), and hope that the outcome is close enough to the expectation.

We employ a standard Bayesian viewpoint to generate an error bar. Suppose, our outcome of the binomial draw is $X = r$. Conditioned on a choice of p, we calculate the probability that $X = r$. This gives a prior on p. Of course, this cannot be done explicitly because of computational issues, but we can use tail bounds for $B(k,p)$ to get appropriate estimates. We use the following theorem of Chernoff [8] (we use notation of Equation 1.4 from [11]) that gives good tail bounds for $B(k,p)$.

THEOREM 5.1 (CHERNOFF). *Suppose $X \sim B(k,p)$. Fix $\alpha \in (0,1)$.*

$$\Pr[X/k \geq \alpha] \leq \exp(-D(\alpha,p)k) \quad \text{if } \alpha > p$$
$$\Pr[X/k \leq \alpha] \leq \exp(-D(\alpha,p)k) \quad \text{if } \alpha < p$$

where $D(a,b) = a\ln(a/b) + (1-a)\ln((1-a)/(1-b))$ (the KL-divergence between Bernoulli distributions with expectation a and b).

Suppose the outcome of $X/k = \alpha$. We can use the Chernoff bound to get a range of likely values of p. Think of $\exp(-D(\alpha,p)k)$ as a function of p. The basic properties of the KL-divergence (and simple algebra) imply that $\exp(-D(\alpha,p)k)$ is a unimodal function with a maximum value of 1 at $p = \alpha$ and a minimum of 0 at $p = 0, 1$. That motivates the following definition.

DEFINITION 5.2. *Fix parameters $k, \alpha, x \in (0,1)$. Then $p_l(k,\alpha,x)$ (lower) and $p_u(k,\alpha,x)$ (upper) are the two unique values of p such that $\exp(-D(\alpha,p)k) = x$.*

With this definition, we can give precise error bars. In other words, given the outcome of a binomial random variable $B(k,p)$, we can give an interval of plausible values (up to any desired confidence δ) for p.

COROLLARY 5.3 (BINOMIAL DISTRIBUTION ERROR BAR). *Fix binomial distribution $B(k,p)$, and $\alpha, \delta \in (0,1)$ Then, for any $p \notin [p_l(k,\alpha,\delta), p_u(k,\alpha,\delta)]$,*

$$\Pr_{X \sim B(k,p)}[X/k = \alpha] \leq \delta$$

How does this relate to our algorithms? Observe that in both Alg. 2 and Alg. 4, the variables $count_i$ are binomial random variables. So we can produce errors bars for $count_i/k$ using the above corollary. The final estimates are of the form $\widehat{C}_i = (count_i/k) \cdot K_i$ ($i \neq 1$, and K_i is some fixed scaling, depending on the algorithm and i). So error bars for $count_i/k$ directly translate to error bars for $\widehat{C}_i$. For $i = 1$ (3-stars), we simply add up the errors (in `3-path-sampler`) for $\widehat{C}_3$, $2\widehat{C}_5$, and $4\widehat{C}_6$.

6. EXPERIMENTAL RESULTS

Preliminaries: We implemented our algorithms in `C` and ran our experiments on a computer equipped with a 2x2.4GHz Intel Xeon processor with 6 cores and 256KB L2 cache (per core), 12MB L3 cache, and 64GB memory. We performed our experiments on 13 graphs from SNAP [43]. In all cases, directionality is ignored, and duplicate edges are omitted.

Tab. 3 has the properties of these graphs, where $|V|$ and $|E|$ are the numbers of vertices and edges, respectively.

Exact counts for the motifs are obtained by a well-tuned enumeration (counts and runtime given in Tab. 3). This algorithm only enumerates the 4-cycle, the chordal-4-cycle, and the 4-clique, and uses direct approaches to get other counts. For convenience, we refer to this the *enumeration code*. We do not get into details, but note that this code processes million edge Amazon networks in only 5 seconds[1]. It uses vertex orderings for speedup, analogous to using degeneracy orderings for triangle enumeration [9, 29]. For getting 3-path sampling estimates, we run both `3-path-sampler` and `centered-sampler` as described earlier, with $k = 200K$. We use the outputs of $\widehat{C}_1, \widehat{C}_2, \widehat{C}_3$ as given by `3-path-sampler`, and $\widehat{C}_4, \widehat{C}_5, \widehat{C}_6$ from `centered-sampler`. The runtimes are in the last column of Tab. 3.

Convergence of estimates: To show convergence, we perform detailed runs on the as-skitter graph. We choose this because it is the most difficult to get accurate estimates, since the cycle-based motif counts are small relative to the graph size. We vary the numbers of samples in increments of 2.5K. For each choice of the number of samples, we perform 50 runs of our algorithm. We plot those results in Fig. 4 for tailed-triangles, chordal-4-cycles, and 4-cliques. (Other patterns are omitted due the space considerations, and had even better convergence.) The output of each run (for a given number of samples) is depicted by a blue dot. For 4-clique counts, we can see the spread of outputs reducing. The figure only goes up to 35K samples. (The convergence is so rapid that at around 50K samples, the spread is impossible to see.)

Accuracy: Fig. 2b presents the relative errors for all 13 graphs and all 6 motifs, using 200K samples for both `3-path-sampler` and `centered-sampler`. *All relative errors are less than 1% in all instances.* As expected the relative errors tend to be larger for the less frequent patterns such as 4-cycles and 4-cliques.

Speedup: Fig. 2a presented the speedups achieved over full enumeration by using our path sampling algorithm. Enumeration for flickr and orkut takes order of a day. Since the motifs counts are in the order of tens of billions, there is no hope of getting any scalability. Our algorithms takes less than a minute (even including I/O) for all these graphs.

The benefit of centered 3-paths: We could simply use the basic 3-path sampling given in `3-path-sampler` to approximate all counts. We compare this approach to our final algorithm that use `centered-sampler` for C_4, C_5, C_6 estimates. Comparisons between the relative errors for C_4, C_5, C_6 are given in Fig. 5. ("Basic" denotes simply using `3-path-sampler`, and "centered" is the main algorithm.) We used 200K samples for both algorithms. Some instances of using `3-path-sampler` give somewhat large errors, and `centered-sampler` really cuts these errors down. It shows the power of centered 3-path sampling.

Error bounds: We use Corollary 5.3 (as explained in §5) to compute 99% confidence error bounds for all of our runs. So, for a single run of our algorithm on a candidate graph, we have a mathematical bound on the error that is solely based on output estimates. (These are critical in the situation where we do not know the true answer, and need confidence that the estimates are accurate.) Fig. 6 shows the accuracy of our error bounds with 99% confidence, so $\delta = 0.01$ in Corollary 5.3. In all cases, the provable bounds on the error are always less than 10% and mostly at most 5%. (We stress that the actual error is much smaller.) To the best of our knowledge, no previous sampling based algorithm for motif counting comes with hard mathematical error bars that are practically reasonable.

[1]This is quite competitive with the best existing numbers in the literature of [22], whose algorithm takes 40 minutes on a 90K autonomous systems graph.

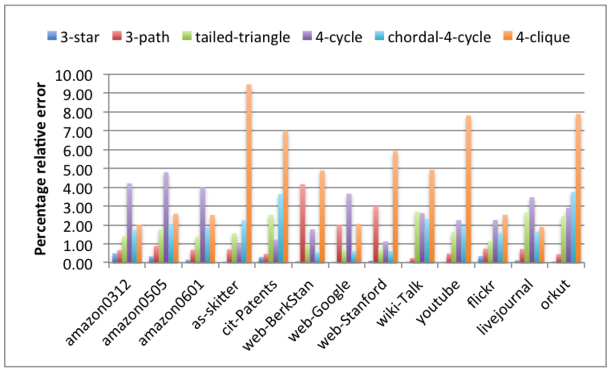


Figure 6: Provable error bounds

Trends in patterns: The most frequent connected induced motif is the 3-star. The least frequent is either the 4-cycle or the 4-clique. We find it intriguing that (among cycle-based motifs) the chordal-4-cycle is the most frequent. (The orkut graph is a notable exception in that 4-cycles are more frequent.) A future direction is to connect these counts with the subgraph frequency approaches of [37].

6.1 Comparison with previous work

Here we present an empirical comparison between our proposed methods and other sampling based algorithms. We focus on color coding methods [19, 4, 42], MCMC based sampling algorithms [5], and edge sampling algorithms [27]. These methods are specifically designed for practicality and scalability. We give a short synopsis of these methods.

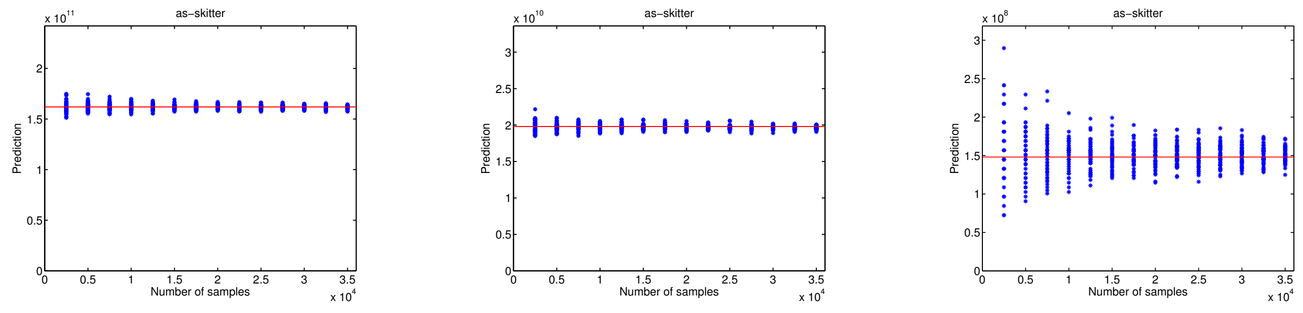
GUISE [5]: This employs a Markov Chain Monte Carlo (MCMC) method to uniformly sample a motif from the space of all (induced) occurrences of motifs of sizes 3, 4, and 5. We use the implementation of GUISE from http://cs.iupui.edu/~alhasan/software.html.

GRAFT [27]: This algorithm samples a set of edges uniformly at random from the input graph and then counts the number of occurrences of each motif that uses any of these sampled edges. These counts can then be scaled appropriately to obtain unbiased estimates. Observe that if edges are sampled with probability 1, this gives an algorithm for exact counting using an edge iteration. For our experiments, we have used our own implementation.

Color Coding [1, 19, 4, 42]: This is a general technique for pattern counting, which samples to prune the enumeration search tree. We randomly color the vertices of the graph and then only enumerate over motifs all of whose vertices have distinct colors. These counts can be appropriately scaled to get unbiased estimates for the true frequencies. For our experiments, we implemented color coding (with 4 colors), and used the same algorithmic techniques that we used for our enumeration algorithm.

Table 3: Exact values of pattern counts and runtimes (in seconds).

Datasets	$\lvert V\rvert$	$\lvert E\rvert$	3-star	3-path	Tailed triangle	4-cycle	Chordal 4-cycle	4-clique	Enum. time	3-path time
amazon0312	4.01E+5	2.35E+6	1.07E+10	8.44E+08	1.90E+08	3.23E+06	1.71E+07	3.98E+06	4.42	0.47
amazon0505	4.10E+5	2.44E+6	1.21E+10	9.63E+08	2.19E+08	3.30E+06	1.91E+07	4.36E+06	4.75	0.48
amazon0601	4.03E+5	2.44E+6	1.11E+10	9.41E+08	2.17E+08	3.22E+06	1.92E+07	4.42E+06	4.74	0.48
as-skitter	1.70E+6	1.11E+7	9.64E+13	8.19E+11	1.62E+11	4.27E+10	1.96E+10	1.49E+08	5128.93	2.7
cit-Patents	3.77E+6	1.65E+7	6.11E+9	6.54E+09	5.52E+08	2.69E+08	6.28E+07	3.50E+06	46.46	3.33
flickr	1.86E+6	1.56E+7	1.90E+13	6.89E+12	1.18E+11	1.18E+11	2.30E+11	2.67E+10	217274.39	2.53
livejournal	5.28E+6	4.87E+7	4.46E+12	1.14E+12	1.26E+11	5.21E+09	1.90E+10	1.14E+10	11894.63	6.86
orkut	3.07E+6	2.24E+8	9.78E+13	1.86E+13	1.51E+12	7.01E+10	4.78E+10	3.22E+09	70966.96	16.24
web-BerkStan	6.85E+5	6.65E+6	3.82E+14	3.14E+10	4.76E+11	2.53E+10	9.86E+10	1.07E+09	6462.56	3.77
web-Google	8.76E+5	4.32E+6	6.50E+11	4.06E+09	6.72E+09	3.80E+07	3.82E+08	3.99E+07	52.29	0.88
web-Stanford	2.82E+5	1.99E+6	2.51E+13	1.28E+10	5.08E+10	4.48E+09	8.60E+09	7.88E+07	831.50	1.76
wiki-Talk	2.39E+6	4.66E+6	1.92E+14	1.17E+12	6.41E+10	9.24E+08	1.03E+09	6.49E+07	1346.76	1.04
youtube	1.16E+6	4.95E+6	5.73E+12	9.15E+10	1.24E+10	2.32E+08	2.22E+08	4.99E+06	141.78	0.61



(a) tailed-triangle (b) chordal-4-cycle (c) 4-clique

Figure 4: Increasing number of samples decreases error: Each blue dot is an output of a run of the algorithm with the number of samples in the x-axis. The red line is the true count.

All these methods are quite general and work for motifs of any size. These algorithms are more general than our approach. But our focus on the specific six subgraphs in Fig. 1 allows for the design of faster and more accurate algorithms, that work better than these generic methods.

For a fair comparison, all running times for previous algorithms are for estimating frequencies of only the cycle-based motifs. (We are being conservative here, since we compare with the entire time for 3-path sampling.) Both GUISE and GRAFT take as input a number of samples. We ran GUISE for 10 million samples, since for fewer samples, the errors were usually around 100%. At 10 million samples, the running time was more than the enumeration cost (usually over 5 times that), so we simply terminated. We increased the number of samples for GRAFT until the errors were within 5% or the running time became larger than 5 times the enumeration cost. For moderately sized graphs, the enumeration running time is typically 2-3 orders of magnitude more than that of 3-path sampling. Color coding takes no parameters, and uses 4 colors.

Our results are summarized in Fig. 7 (running time comparison) and Tab. 4 (accuracy comparison). For ease of presentation, we focus on 4 graphs: as-skitter, cit-patents, web-Stanford, and wiki-Talk. These range from a million to 10 million edges. Our results were similar on other graphs. We present results for a single run of each algorithm. (We ran for numerous iterations, and all results were consistent.) In summary, our algorithm is many orders of magnitude faster and more accurate than these methods. We stress that these results are consistent with the literature, where previous methods were only employed on graphs with about 100K edges.

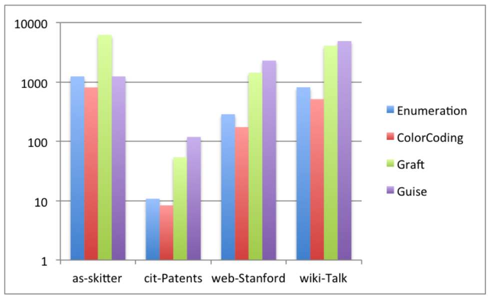
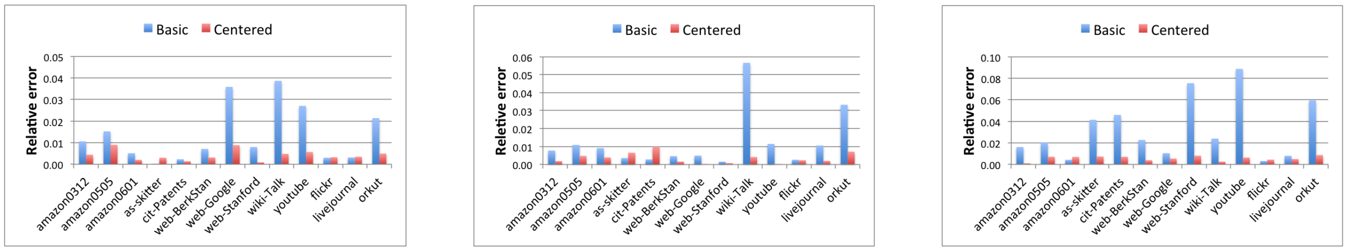


Figure 7: Runtimes of the existing algorithms normalized with respect to runtimes of 3-path sampling algorithm

Scalability and speed. Previous works either leverage Map-Reduce clusters or max out at a million (or so) edges. Our 3-path sampler runs on a single commodity machine with 64GB memory, and easily handles graphs with more than a hundred million edges.
Running time comparisons are in Fig. 7, where for each graph, we normalize by the running time of 3-path sampler. This figure is limited to only moderate sized graphs due to large runtimes of the other algorithms. At about a million edges, 3-path sampling is already hundreds of times faster than other algorithms. (As we explain in the next section and



(a) 4-cycle (b) Chordal 4-cycle (c) 4-clique

Figure 5: Comparing the accuracy of estimations using basic 3-path sampling and centered 3-path sampling

Tab. 4, the accuracy of 3-path sampling is much better than GRAFT and GUISE, and comparable to color coding.) We note that our enumeration code is actually competitive with existing sampling algorithms. Color coding is consistently (a little) faster than enumeration. GUISE and GRAFT do not yield accurate approximations even when run for many multiples of the enumeration time. We believe that this is an issue of scale, since previous algorithm were not run for graphs with many millions of edges.

We provide some explanations for these runtimes. GRAFT provides a speedup over edge iteration enumeration procedures. However, edge-iteration algorithms are not the best methods for enumeration, as is well-known for triangle enumeration [29, 3]. Enumeration using vertex orderings is significantly faster. The same principles to apply to counting 4-vertex patterns. For small sizes (100K edges or less), GRAFT may give improvements over the best enumeration, but this is not true for larger sizes.

GUISE estimates the relative frequencies of the motifs, not the exact frequencies. GUISE has to perform relatively long random walks before it can obtain samples from the stationary distribution. This limits the number of samples that can be made. More problematically, the universe from which GUISE samples from uniformly is prohibitively large. In particular, focusing only on 4-vertex motifs, we see that (say for as-Skitter), the fraction of cliques in the universe of 4-vertex motifs is less than 10^{-6}. This means that roughly a million samples are needed to witness a single 4-clique, and the square of a million samples to estimate accurately.

Color coding is hindered by the sheer size of the output. Color coding cuts down the set of motifs by enumerating only polychromatic motifs (i.e. all vertices with distinct colors), but this set is still quite large. In particular, for 4-vertex motifs, the reduction of the output is only of the order of one-tenth (or 3/32 to be precise – the probability that four vertices of a motif all get different colors when using 4 colors). This means enumerating over a billion motifs in as-skitter, for example. There is also the extra overhead of actually searching through neighborhoods to perform this enumeration. We do see that color coding provides some benefit over enumeration, and is probably the best algorithm for 4-vertex pattern counting among previous work.

Accuracy. We present the accuracies of the various algorithms in Tab. 4. We focus on 4-clique counts for brevity. Even after running for times more than enumeration, GRAFT and GUISE give low accuracies. GUISE generally fails to even find a 4-clique, and GRAFT has not processed enough samples to converge. Color coding is extremely accurate, and 3-path sampling is competitive. But the running time of color coding is a hundred to thousand times that of 3-path sampling on these instances.

Table 4: Relative error in 4-clique count. For GUISE, we choose 10M samples. For GRAFT, we report the errors after running the algorithm for up to 5 times the enumeration time.

	3-path	Color Coding	Graft	Guise
as-skitter	0.7	0.1	95	99
cit-Patents	0.7	0.4	78	54
web-Stanford	0.8	0.8	29	99
wiki-Talk	0.3	0.5	5	99

7. CONCLUSIONS AND FUTURE WORK

We get accurate results for all 4-vertex motif frequencies on a large number of graphs, and believe this is useful for motif analyses. Previous work usually focuses on a small, specific set of larger motifs [19, 4, 42], or gives coarser approximations for more motifs [5]. It is natural to ask if we can extend this sampling scheme further to estimate counts of 5-vertex (or even higher order) motifs.

8. REFERENCES

[1] N. Alon, R. Yuster, and U. Zwick. Color-coding: A new method for finding simple paths, cycles and other small subgraphs within large graphs. In *Proceedings of the Twenty-sixth Annual ACM Symposium on Theory of Computing*, STOC '94, pages 326–335, New York, NY, USA, 1994. ACM.

[2] L. Becchetti, P. Boldi, C. Castillo, and A. Gionis. Efficient semi-streaming algorithms for local triangle counting in massive graphs. In *KDD'08*, pages 16–24, 2008.

[3] J. W. Berry, L. K. Fostvedt, D. J. Nordman, C. A. Phillips, C. Seshadhri, and A. G. Wilson. Why do simple algorithms for triangle enumeration work in the real world? In *Proceedings of the 5th Conference on Innovations in Theoretical Computer Science*, ITCS '14, pages 225–234, New York, NY, USA, 2014. ACM.

[4] N. Betzler, R. van Bevern, M. R. Fellows, C. Komusiewicz, and R. Niedermeier. Parameterized algorithmics for finding connected motifs in biological networks. *IEEE/ACM Trans. Comput. Biology Bioinform.*, 8(5):1296–1308, 2011.

[5] M. Bhuiyan, M. Rahman, M. Rahman, and M. A. Hasan. Guise: Uniform sampling of graphlets for large

graph analysis. In *Proceedings of International Conference on Data Mining*, pages 91–100, 2012.

[6] M. A. Bhuiyan and M. A. Hasan. Mirage: An iterative mapreduce based frequent subgraph mining algorithm. Technical report, arXiv, 2013. http://arxiv.org/pdf/1307.5894.pdf.

[7] R. Burt. Structural holes and good ideas. *American Journal of Sociology*, 110(2):349–399, 2004.

[8] H. Chernoff. A measure of asymptotic efficiency for test of a hypothesis based on the sum of observations. *Annals of mathematical statistics*, 23:493–507, 1952.

[9] N. Chiba and T. Nishizeki. Arboricity and subgraph listing algorithms. *SIAM J. Comput.*, 14:210–223, 1985.

[10] J. Coleman. Social capital in the creation of human capital. *American Journal of Sociology*, 94:S95–S120, 1988.

[11] D. Dubhashi and A. Panconesi. *Concentration of Measure for the Analysis of Randomized Algorithms*. Cambridge University Press, 2009.

[12] N. Durak, A. Pinar, T. G. Kolda, and C. Seshadhri. Degree relations of triangles in real-world networks and graph models. In *CIKM'12*, 2012.

[13] G. Fagiolo. Clustering in complex directed networks. *Phys. Rev. E*, 76:026107, Aug 2007.

[14] K. Faust. A puzzle concerning triads in social networks: Graph constraints and the triad census. *Social Networks*, 32(3):221–233, 2010.

[15] M. Gonen and Y. Shavitt. Approximating the number of network motifs. *Internet Mathematics*, 6(3):349–372, 2009.

[16] D. Hales and S. Arteconi. Motifs in evolving cooperative networks look like protein structure networks. *NHM*, 3(2):239–249, 2008.

[17] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. American Statistical Association*, 58:13–30, 1963.

[18] P. Holland and S. Leinhardt. A method for detecting structure in sociometric data. *American Journal of Sociology*, 76:492–513, 1970.

[19] F. Hormozdiari, P. Berenbrink, N. Prÿulj, and S. C. Sahinalp. Not all scale-free networks are born equal: The role of the seed graph in ppi network evolution. *PLoS Computational Biology*, 118, 2007.

[20] S. Itzkovitz, R. Levitt, N. Kashtan, R. Milo, M. Itzkovitz, and U. Alon. Coarse-graining and self-dissimilarity of complex networks. 71(016127), January 2005.

[21] T. G. Kolda, A. Pinar, T. Plantenga, C. Seshadhri, and C. Task. Counting triangles in massive graphs with MapReduce. *SIAM Journal of Scientific Computing*, 2013. To appear.

[22] D. Marcus and Y. Shavitt. Efficient counting of network motifs. In *ICDCS Workshops*, pages 92–98. IEEE Computer Society, 2010.

[23] R. Milo, S. Shen-Orr, S. Itzkovitz, N. Kashtan, D. Chklovskii, and U. Alon. Network motifs: Simple building blocks of complex networks. *Science*, 298(5594):824–827, 2002.

[24] T. Plantenga. Inexact subgraph isomorphism in mapreduce. *Journal of Parallel and Distributed Computing*, (0), 2012.

[25] A. Portes. Social capital: Its origins and applications in modern sociology. *Annual Review of Sociology*, 24(1):1–24, 1998.

[26] N. Przulj, D. G. Corneil, and I. Jurisica. Modeling interactome: scale-free or geometric?. *Bioinformatics*, 20(18):3508–3515, 2004.

[27] M. Rahman, M. A. Bhuiyan, and M. A. Hasan. Graft: An efficient graphlet counting method for large graph analysis. *IEEE Transactions on Knowledge and Data Engineering*, PP(99), 2014.

[28] T. Schank and D. Wagner. Approximating clustering coefficient and transitivity. *Journal of Graph Algorithms and Applications*, 9:265–275, 2005.

[29] T. Schank and D. Wagner. Finding, counting and listing all triangles in large graphs, an experimental study. In *Experimental and Efficient Algorithms*, pages 606–609. Springer Berlin / Heidelberg, 2005.

[30] C. Seshadhri, T. G. Kolda, and A. Pinar. Community structure and scale-free collections of Erdös-Rényi graphs. *Physical Review E*, 85(5):056109, May 2012.

[31] C. Seshadhri, A. Pinar, and T. G. Kolda. Fast triangle counting through wedge sampling. In *Proceedings of the SIAM Conference on Data Mining*, 2013.

[32] S. Son, A. Kang, H. Kim, T. Kwon, J. Park, and H. Kim. Analysis of context dependence in social interaction networks of a massively multiplayer online role-playing game. *PLoS ONE*, 7(4):e33918, 04 2012.

[33] M. Szell and S. Thurner. Measuring social dynamics in a massive multiplayer online game. *Social Networks*, 32:313–329, 2010.

[34] C. Tsourakakis, P. Drineas, E. Michelakis, I. Koutis, and C. Faloutsos. Spectral counting of triangles in power-law networks via element-wise sparsification. In *ASONAM'09*, pages 66–71, 2009.

[35] C. Tsourakakis, M. N. Kolountzakis, and G. Miller. Triangle sparsifiers. *J. Graph Algorithms and Applications*, 15:703–726, 2011.

[36] C. E. Tsourakakis, U. Kang, G. L. Miller, and C. Faloutsos. Doulion: counting triangles in massive graphs with a coin. In *Knowledge Data and Discovery (KDD)*, pages 837–846, 2009.

[37] J. Ugander, L. Backstrom, and J. M. Kleinberg. Subgraph frequencies: mapping the empirical and extremal geography of large graph collections. In D. Schwabe, V. A. F. Almeida, H. Glaser, R. A. Baeza-Yates, and S. B. Moon, editors, *WWW*, pages 1307–1318. International World Wide Web Conferences Steering Committee / ACM, 2013.

[38] A. Walker. Walker, a. j. (1977). "an efficient method for generating discrete random variables with general distributions. *ACM Transactions on Mathematical Software (TOMS)*, 3(3):253–256, 1977.

[39] D. Watts and S. Strogatz. Collective dynamics of 'small-world' networks. *Nature*, 393:440–442, 1998.

[40] B. Welles, A. Van Devender, and N. Contractor. Is a friend a friend?: Investigating the structure of friendship networks in virtual worlds. In *CHI-EA'10*, pages 4027–4032, 2010.

[41] E. Wong, B. Baur, S. Quader, and C.-H. Huang. Biological network motif detection: principles and practice. *Briefings in Bioinformatics*, 13(2):202–215, 2012.

[42] Z. Zhao, G. Wang, A. Butt, M. Khan, V. S. A. Kumar, and M. Marathe. Sahad: Subgraph analysis in massive networks using hadoop. In *Proceedings of International Parallel and Distributed Processing Symposium (IPDPS)*, pages 390–401, 2012.

[43] Stanford Network Analysis Project (SNAP). Available at http://snap.stanford.edu/.