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Some variational principles for Z-eigenvalues of nonnegative tensors

K.C. Chang [a], K.J. Pearson [b], Tan Zhang [b,*]

[a] LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China
[b] Department of Mathematics and Statistics, Murray State University, Murray, KY 42071, USA

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ABSTRACT

Many important spectral properties of nonnegative matrices have recently been successfully extended to higher order nonnegative tensors; for example, see (Chang et al., 2008, 2011; Friedland et al., in press; Lim, 2005; Liu et al., 2010; Ng et al., 2010; Qi et al., 2007; Yang and Yang, 2010) [2,3,9,17,23,24,27,28]. However, most of these results focus on the H-eigenvalues introduced by Qi (2005, 2007) [25,26]. The key results of this paper reveal some similarities as well as some crucial differences between Z-eigenvalues and H-eigenvalues of a nonnegative tensor. In particular, neither the positive Z-eigenvalue nor the associated positive Z-eigenvector of an irreducible nonnegative tensor has to be unique in general as demonstrated by Example 2.7. Furthermore, the Collatz type min–max characterizations of the largest positive Z-eigenvalue of an irreducible nonnegative tensor is only partially true in general as seen in Theorem 4.7 and Example 4.6.

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1. Introduction

Let $\mathbb{R}$ be the real field, we consider an m-order n-dimensional tensor $\mathcal{A}$ consisting of n^m entries in $\mathbb{R}$:

$$\mathcal{A} = (a_{i_1 \cdots i_m}), \quad a_{i_1 \cdots i_m} \in \mathbb{R}, \quad 1 \leq i_1, \ldots, i_m \leq n.$$

An m-order n dimensional tensor $\mathcal{A}$ is called nonnegative, denoted $\mathcal{A} \geq 0$, if $a_{i_1 \cdots i_m} \geq 0$.

* Corresponding author. Tel.: +1 270 809 3712.
E-mail addresses: kcchang@math.pku.edu.cn (K.C. Chang), kpearson@murraystate.edu (K.J. Pearson), tzhang@murraystate.edu (T. Zhang).

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A higher order tensor is a multilinear map, which is a natural generalization of a matrix; a matrix is simply an order two tensor. We shall denote the set of all m-order n-dimensional tensors by $\mathbb{R}^{[m,n]}$ and the set of all nonnegative m-order n-dimensional tensors by $\mathbb{R}_+^{[m,n]}$ throughout the rest of this paper.

To an n-vector $x = (x_1, \cdots, x_n)$, real or complex, we define the n-vector:

$$\mathcal{A}x^{m-1} := \left(\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \right)_{1\le i\le n}$$

and the n-vector $x^{[m-1]} := (x_1^{m-1}, \cdots, x_n^{m-1})$.

The following were first introduced and studied by Qi and Lim [17,25–27].

Definition 1.1. Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an eigenvalue-eigenvector (or simply eigenpair) of $\mathcal{A}$ if they satisfy the equation

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{1}$$

We call (λ, x) an H-eigenpair if they are both real.

Definition 1.2. Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. A pair $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is called an E-eigenvalue and E-eigenvector (or simply E-eigenpair) of $\mathcal{A}$ if they satisfy the equation

$$\begin{cases} \mathcal{A}x^{m-1} = \lambda x, \\ x^T x = 1 \end{cases} \tag{2}$$

We call (λ, x) a Z-eigenpair if they are both real.

Both H-eigenvalues and Z-eigenvalues of a given tensor have found vibrant new applications in numerical multilinear algebra, image processing, higher order Markov chains, and spectral hypergraph theory. Although both of these eigenvalue problems for tensors are nonlinear, their chief difference lies in that the H-eigenvalue problem (1) is equivalent to finding nontrivial solutions of a system of homogeneous polynomial equations of the same degree in several variables, whereas the Z-eigenvalue problem (2) is equivalent to finding nontrivial solutions of a system of inhomogeneous polynomial equations in several variables.

We further introduce the following.

Definition 1.3. [2,17] A tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in \mathbb{R}^{[m,n]}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, \ldots, n\}$ such that

$$a_{i_1\cdots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \ldots, i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

The above definition for irreducibility has been used extensively in the literature; for example, see [1,2,23,24,28]. In some articles, the alternative notion of weak irreducibility is used; for example, see [9].

There is abundant literature on the development of H-eigenvalues for nonnegative tensors; for example, see [2,3,9,17,23,24,27,28]. In particular, the largest positive H-eigenvalue of a nonnegative tensor is related to measuring higher order connectivity in linked objects [18] and hypergraphs [8]. Many effective algorithms for finding the largest positive H-eigenvalue and the corresponding positive eigenvector of a nonnegative tensor have been implemented; for more detailed discussions, see [3,9,23,24].

Equally important, E/Z-eigenvalues play a fundamental role in best rank-one approximation. The best rank-one approximation of higher order tensors has numerous applications in engineering and higher order statistics, such as Statistical Data Analysis. We refer the interested reader to [7,14,29] for a more systematic treatment on the analysis, algorithms, and applications of the least-squares approximation of a tensor by a tensor of low multilinear rank or a tensor of low rank, e.g. in the canonical polyadic decomposition, canonical decomposition, or parallel factor decomposition.

In a series of their recent work [19–22], Ng et al. discovered intriguing new connections of the Z-eigenvalue problem to the transition probability tensors of higher order Markov chains. They propose a framework (HAR) that can be used in multi-relational data mining. Some other related work in this new front have been conducted in [4] and [11]. In [4,11,19,22], an alternative normalization of Z-eigenvalue problem is employed (using the ℓ_1-norm instead of the conventional ℓ_2-norm); this amounts to a rescaling of both the eigenvalue as well as the corresponding eigenvector (cf. Theorem 1.3 [4]).

Using Z-eigenvalues, Qi et al. [10,16] have recently extended the notion and results regarding the algebraic connectivity of a graph in spectral graph theory to k-uniform hypergraphs.

It is also worth mentioning that Qi et al. [12] have found a new exciting application of some of our main results presented here in quantum entanglement problem. In particular, based on the Z-spectral radius of a nonnegative tensor first introduced in Section 3 of this paper, they establish a surprising connection to the geometric measure of quantum entanglement for a symmetric pure state with nonnegative amplitudes.

While the spectral theory of H-eigenvalues for nonnegative tensors is relatively complete and more mature, the spectral theory of Z-eigenvalues for nonnegative tensors is still underdeveloped and in its early stage. With this in mind, we endeavor to highlight the similarities, but more importantly, the differences one may encounter when dealing with Z-eigenvalue problems in general.

Our paper is organized as follows. In Section 2, we review the Perron–Frobenius Theorems for both H- and Z-eigenvalues of nonnegative tensors from the previous work of Chang et al. [2]. Example 2.7 demonstrates the non-uniqueness of positive Z-eigenvalues and corresponding positive Z-eigenvectors of a nonnegative irreducible tensor, which is strikingly different from H-eigenvalues. In Section 3, we conduct a systematic investigation of the Z-spectrum $\mathcal{Z}(\mathcal{A})$ and the nonnegative Z-spectrum $\Lambda(\mathcal{A})$ of a (nonnegative) tensor $\mathcal{A}$. It will be seen that quite different from the H-eigenvalues, $\mathcal{Z}(\mathcal{A})$ may be an infinite set. However, Theorem 3.7 establishes the compactness of $\mathcal{Z}(\mathcal{A})$. Next, we review the class of weakly symmetric tensors and show that for a weakly symmetric tensor $\mathcal{A}$, $\mathcal{Z}(\mathcal{A})$ must be a finite set; this is Proposition 3.10. Consequently, by specializing in the subclass of weakly symmetric nonnegative tensors, we obtain Theorem 3.11, which asserts the equalities among the $\mathcal{Z}$-spectral radius (denoted $\varrho(\mathcal{A})$), the maximum of the function $f_{\mathcal{A}}$ over the unit sphere (denoted $\bar{\lambda}$), and the maximum of the nonnegative $\mathcal{Z}$-spectrum $\Lambda(\mathcal{A})$ (denoted λ^*). In Section 4, we study the max–min characterization of λ^*. Similar to H-eigenvalues, the irreducibility of tensors is imposed. In particular, we shall prove:

$$\varrho(\mathcal{A}) = \lambda^* = \overline{\lambda} = \max_{x\in P\cap S^{n-1}} \min_{1\le i\le n} \frac{(\mathcal{A}x^{m-1})_i}{x_i},$$

under the assumptions that $\mathcal{A}$ is nonnegative, weakly symmetric, and irreducible. Examples are given to illustrate that all these assumptions cannot be avoided. In Section 5, we adapt the Shifted Symmetric Higher-Order Power Method (SS-HOPM) proposed by Kolda and Mayo [13] to compute some numerical examples.

2. The Perron–Frobenius Theorem for nonnegative tensors

In this paper, we will mostly be working with nonnegative tensors. One of the pinnacles of the theory of nonnegative matrices is the classical Perron–Frobenius Theorem. The idea of extending powerful results regarding the spectral properties from the realm of nonnegative matrices to the higher order nonnegative tensors setting can be traced back to Lim [17] and subsequently carried out by a number of research teams worldwide; for example, see [2,3,8,9,23,24,28]. To make the comparisons between H- and Z-eigenvalues more transparent, we list some recent progress in this area below, beginning with H-eigenvalues.

Theorem 2.1. (cf. Theorem 1.3 in [2]) *If $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$, then there exists $\lambda_0 \geq 0$ and a nonnegative vector $x_0 \neq 0$ such that $\mathcal{A}x_0^{m-1} = \lambda_0 x_0^{[m-1]}$. In particular, λ_0 is a nonnegative eigenvalue in terms of Definition 1.1.*

Theorem 2.2. (cf. Theorem 1.4 in [2]) *If $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is irreducible, then the pair (λ_0, x_0) in the previous theorem satisfy:*

(1) The H-eigenvalue λ_0 is positive.
(2) The H-eigenvector x_0 is positive, i.e. all components of x_0 are positive.
(3) If λ is an eigenvalue with nonnegative eigenvector, then $\lambda = \lambda_0$. Moreover, the nonnegative H-eigenvector is unique up to a multiplicative constant.
(4) If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leq \lambda_0$.

The spectral radius of a tensor is defined as follows in [28].

Definition 2.3. Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. We define the (H)-spectral radius of $\mathcal{A}$, denoted $\rho(\mathcal{A})$, to be $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } \mathcal{A}\}$, where $|\lambda|$ denotes the modulus of λ.

It is established, for instance in [28], that $\rho(\mathcal{A})$ is itself the largest positive H-eigenvalue of a nonnegative tensor $\mathcal{A}$.

Let $P = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, 1 \leq i \leq n\}$ denote the positive cone in $\mathbb{R}^n$ and $P^\circ = \{(x_1, \ldots, x_n) \in P \mid x_i > 0, 1 \leq i \leq n\}$ its interior. We have the following min–max characterizations of $\rho(\mathcal{A})$.

Theorem 2.4. (cf. Theorem 4.2 in [2]) *If $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is irreducible, then*

$$\min_{x\in P^\circ} \max_{\{i|x_i>0\}} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}} = \rho(\mathcal{A}) = \max_{x\in P^\circ} \min_{\{i|x_i>0\}} \frac{(\mathcal{A}x^{m-1})_i}{x_i^{m-1}}. \tag{3}$$

In addition, from a practical viewpoint, one can effectively implement the analog of the power method to numerically compute $\rho(\mathcal{A})$, provided $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is irreducible. For a more systematic treatment of this topic, we refer the interested readers to [3,9,24] for details.

Similar to H-eigenvalues, we also have the following statements for Z-eigenvalues:

Theorem 2.5. *If $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$, then there exists a Z-eigenvalue $\lambda_0 \geq 0$ and a nonnegative Z-eigenvector $x_0 \neq 0$ of $\mathcal{A}$ such that $\mathcal{A}x_0^{m-1} = \lambda_0 x_0$.*

Theorem 2.6. *If $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is irreducible, then the pair (λ_0, x_0) in Theorem 2.5 satisfy:*

(1) The eigenvalue λ_0 is positive.
(2) The eigenvector x_0 is positive, i.e. all components of x_0 are positive.

In [2], a more general version of Theorems 2.1, 2.2, 2.5, and 2.6 was established. To avoid detouring from our current goal, we shall present the proofs of Theorems 2.5 and 2.6 in the appendix. These proofs are modified from our original proof given in Section 2 [2] to specifically address Z-eigenvalues.

However, unlike H-eigenpairs for a nonnegative irreducible tensor $\mathcal{A}$, neither the positive Z-eigenvalue nor the associated positive Z-eigenvector of $\mathcal{A}$ has to be unique in general. This has been pointed out by a counterexample in the Errata of [2], which is similar to the following.

Example 2.7. Let $\mathcal{A} \in \mathbb{R}_+^{[4,2]}$ be defined by

$$a_{1111} = a_{2222} = \frac{4}{\sqrt{3}}, \quad a_{1112} = a_{1121} = a_{1211} = a_{2111} = 1,$$
$$a_{1222} = a_{2122} = a_{2212} = a_{2221} = 1, \quad \text{and} \quad a_{ijkl} = 0 \quad \text{elsewhere.}$$

It is not difficult to check $\mathcal{A}$ is irreducible. It is straightforward to compute, whose detailed calculation will be carried out in the subsequent section hence omitted here, that $\mathcal{A}$ has two positive Z-eigenvalues:

$\lambda_0 = 2 + \frac{2}{\sqrt{3}} \approx 3.1547$ with corresponding positive Z-eigenvector $x_0 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;

$\lambda_1 = \frac{11}{2\sqrt{3}} \approx 3.1754$ with corresponding positive Z-eigenvectors $x_1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $x_2 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

It was shown in [5], for $m > 2$, the degree of the E-characteristic polynomial $\psi_{\mathcal{A}}(\lambda)$ of a generic tensor $\mathcal{A}$ is

$$d_E = ((m-1)^n - 1)/(m-2) = (m-1)^{n-1} + (m-1)^{n-2} + \cdots + (m-1) + 1.$$

It was shown in [15], the E-characteristic polynomial, hence Z-eigenvalues, are in fact invariant under the action of the orthogonal group. Hence, many results we prove for Z-eigenvalues of nonnegative tensors remain valid for a broader class of tensors which are not necessarily nonnegative themselves but are orthogonally similar to nonnegative tensors. Unfortunately, H-eigenvalues are not orthogonally invariant.

Consequently, from an invariance theory perspective, Z-eigenvalues seem more desirable. However, as we shall see in the latter part of this paper, for nonnegative tensors, Z-eigenvalues lack certain important minimax characterization such as the Collatz type Theorem 2.4.

3. The nonnegative Z-spectrum of $\mathcal{A}$

The main focus of this section is to study the nonnegative Z-eigenvalues of an m-order n-dimensional tensor $\mathcal{A}$ if $\mathcal{A}$ is a nonnegative tensor or $\mathcal{A}$ is a weakly symmetric tensor. Some important characterizations of the largest Z-eigenvalue of $\mathcal{A}$ are established at the end of this section if $\mathcal{A}$ is both nonnegative and weakly symmetric. We begin with some definitions.

Similar to H-eigenvalues, we may define the Z-spectrum of $\mathcal{A}$ as follows.

Definition 3.1. Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. We define the Z-spectrum of $\mathcal{A}$, denoted $\mathcal{Z}(\mathcal{A})$ to be the set of all Z-eigenvalues of $\mathcal{A}$. Assume $\mathcal{Z}(\mathcal{A}) \neq \emptyset$, then the Z-spectral radius of $\mathcal{A}$, denoted $\varrho(\mathcal{A})$, is defined as $\varrho(\mathcal{A}) := \sup\{|\lambda| \mid \lambda \in \mathcal{Z}(\mathcal{A})\}$.

Similar to H-eigenvalues, the set $\mathcal{Z}(\mathcal{A})$ may be the empty set; see the following:

Example 3.2. Let $n = 2$. Let $\mathcal{A}$ be given by

$$a_{12\ldots2} = 1, \quad a_{21\ldots1} = -1, \quad \text{and} \quad a_{i_1\cdots i_m} = 0 \quad \text{elsewhere.}$$

Then the Z-eigenvalue problem is to solve

$$\begin{cases} x_2^{m-1} = \lambda x_1, \\ -x_1^{m-1} = \lambda x_2, \end{cases}$$

which by elimination, yields no nonzero real solution if m is even.

We now justify $\varrho(\mathcal{A})$ is well defined. The proof is a routine exercise and resembles that of $\rho(\mathcal{A})$.

Proposition 3.3. *Let* $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$. *Then* $\varrho(\mathcal{A}) \leq \sqrt{n}\max_{1\leq i\leq n}\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m}$.

Proof. Since $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$, we have $\mathcal{Z}(\mathcal{A}) \neq \emptyset$ by Theorem 2.5. Let (λ, x) be a Z-eigenpair of $\mathcal{A}$. Since $x = (x_1, \cdots, x_n)$ with $x_1^2 + \cdots + x_n^2 = 1$, there exists an index $i_0 \in \{1, \ldots, n\}$ such that $|x_{i_0}| \geq \frac{1}{\sqrt{n}}$. We also have:

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{i_0 i_2\cdots i_m} x_{i_2}\cdots x_{i_n} = \lambda x_{i_0}.$$

Since $|x_i| \leqslant 1$, we have:

$$\frac{1}{\sqrt{n}}|\lambda| \leq |\lambda||x_{i_0}| \leq \sum_{i_2,\ldots,i_m=1}^{n} a_{i_0 i_2\cdots i_m};$$

our assertion now follows. □

Indeed, the assumption on the non-negativity of $\mathcal{A}$ is too strong; as long as $\mathcal{Z}(\mathcal{A}) \neq \emptyset$, we have $\varrho(\mathcal{A}) \leq \sqrt{n}\max_{1\leq i\leq n}\sum_{i_2,\ldots,i_m=1}^{n} |a_{ii_2\cdots i_m}|$.

In contrast to the H-spectral radius $\rho(\mathcal{A})$, the Z-spectral radius $\varrho(\mathcal{A})$ of a nonnegative tensor $\mathcal{A}$ may not be itself a positive Z-eigenvalue of $\mathcal{A}$. This is demonstrated by the following example.

Example 3.4. Let $\mathcal{A} \in \mathbb{R}_+^{[4,2]}$ be defined by

$$\begin{aligned} &a_{1112} = 30, \quad a_{1212} = 1, \quad a_{1222} = 1, \quad a_{2111} = 6, \\ &a_{2112} = 13, \quad a_{2122} = 37, \quad \text{and} \quad a_{ijkl} = 0 \quad \text{elsewhere}. \end{aligned}$$

The Z-eigenvalue problem is to solve:

$$\begin{cases} 30x_1^2x_2 + x_1x_2^2 + x_2^3 = \lambda x_1, \\ 6x_1^3 + 13x_1^2x_2 + 37x_1x_2^2 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

Note $\mathcal{A}$ is irreducible and we calculate the three Z-eigenpairs of $\mathcal{A}$ to be:

1. The Z-eigenvalue $\lambda_1 = \frac{63}{5}$ with its corresponding Z-eigenvectors $\pm\left(\frac{\sqrt{10}}{10}, \frac{3\sqrt{10}}{10}\right)$.
2. The Z-eigenvalue $\lambda_2 = -\frac{64}{5}$ with its corresponding Z-eigenvectors $\left(\pm\frac{\sqrt{5}}{5}, \mp\frac{2\sqrt{5}}{5}\right)$.
3. The Z-eigenvalue $\lambda_3 = -15$ with its corresponding Z-eigenvectors $\left(\pm\frac{\sqrt{2}}{2}, \mp\frac{\sqrt{2}}{2}\right)$.

In this case, $\varrho(\mathcal{A}) = |\lambda_3| = 15$, but 15 is not a Z-eigenvalue of $\mathcal{A}$. To see this, using the computational commutative algebra system CoCoA[6], we consider the ideal generated by

$$\{30x_1^2x_2 + x_1x_2^2 + x_3^2 - \lambda x_1,\ 6x_1^3 + 13x_1^2x_2 + 37x_1x_2^2 - \lambda x_2,\ x_1^2 + x_2^2 - 1\}.$$

By computing the elimination ideal via eliminating the variables x_1 and x_2, we come up with the E-characteristic polynomial $\psi_{\mathcal{A}}$ of $\mathcal{A}$:

$$\psi_{\mathcal{A}}(\lambda) = 25\lambda^4 + 755\lambda^3 + 1743\lambda^2 - 119835\lambda - 907200;$$

whose four real zeros are $\lambda_1 = \frac{63}{5}$, $\lambda_2 = -\frac{64}{5}$, and $\lambda_3 = -15$ with algebraic multiplicity two.

Definition 3.5. Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$. We define the nonnegative spectrum of $\mathcal{A}$, denoted $\Lambda(\mathcal{A})$ to be the set of all $\lambda \geq 0$ such that there exists $x \in P \cap S^{n-1}$ satisfying $\mathcal{A}x^{m-1} = \lambda x$, where S^{n-1} is the standard unit sphere in $\mathbb{R}^n$.

It is important to notice the set $\Lambda(\mathcal{A})$ (thus $\mathcal{Z}(\mathcal{A})$) is not necessarily a finite set in general. This is a very notable distinction from the H-spectrum, since the set of all H-eigenvalues of $\mathcal{A}$ is always finite. This very distinction stems from the homogeneous versus inhomogeneous systems involved in solving the two types of eigenvalue problems. We consider the following example.

Example 3.6. Let $\mathcal{A} \in \mathbb{R}_+^{[4,2]}$ be defined by

$$a_{1112} = a_{2122} = 2 \quad \text{and} \quad a_{ijkl} = 0 \quad \text{elsewhere.}$$

The Z-eigenvalue problem is to solve:

$$\begin{cases} 2x_1^2 x_2 = \lambda x_1, \\ 2x_1 x_2^2 = \lambda x_2, \\ x_1^2 + x_2^2 = 1. \end{cases}$$

Let $(x_1, x_2) \in P \cap S^1$. If $x_1 = 0$ (implying $x_2 = 1$) or $x_2 = 0$ (implying $x_1 = 1$), then $\lambda = 0$. Let $(x_1, x_2) \in P^\circ \cap S^1$ satisfy $0 < 2x_1x_2 \leq 1$. By taking $\lambda = 2x_1x_2$, we see $(\lambda, (x_1, x_2))$ is a Z-eigenpair. Hence, $\Lambda(\mathcal{A}) = [0, 1]$, the whole closed interval.

Theorem 3.7. *Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$. Then $\emptyset \neq \Lambda(\mathcal{A}) \subset \mathcal{Z}(\mathcal{A}) \subset \mathbb{R}$ is a compact subset.*

Proof. We proceed in the following steps. First, the fact that $\Lambda(\mathcal{A})$ is nonempty follows from Theorem 2.5. Second, the boundedness of both $\mathcal{Z}(\mathcal{A})$ and $\Lambda(\mathcal{A})$ follow from Proposition 3.3. Third, the fact that $\mathcal{Z}(\mathcal{A})$ and $\Lambda(\mathcal{A})$ are both closed follow from the compactness of the unit sphere S^{n-1} and the continuity of the mapping $x \mapsto \mathcal{A}x^{m-1}$. To be more explicit, we shall only present the details to show $\Lambda(\mathcal{A})$ is closed since a similar argument shows $\mathcal{Z}(\mathcal{A})$ is also closed. Choose a sequence $\{\lambda_k\} \subset \Lambda(\mathcal{A})$ and a sequence $\{x_k\} \subset P \cap S^{n-1}$ with $\mathcal{A}x_k^{m-1} = \lambda_k x_k$ such that $\lambda_k \to \tilde{\lambda}$ as $k \to \infty$. We must show $\tilde{\lambda} \in \Lambda(\mathcal{A})$. Clearly, such $\tilde{\lambda} \geq 0$. Since $\{x_k\} \subset S^{n-1}$, there exists a convergent subsequence of $\{x_k\}$; without loss of generality, we still denote it by $\{x_k\}$, such that $x_k \to \tilde{x} \in P \cap S^{n-1}$ as $k \to \infty$. By continuity, we see that $\mathcal{A}x_k^{m-1} \to \mathcal{A}\tilde{x}^{m-1}$ and $\lambda_k x_k \to \tilde{\lambda}\tilde{x}$ as $k \to \infty$; hence, $\mathcal{A}\tilde{x}^{m-1} = \tilde{\lambda}\tilde{x}$, i.e. $\tilde{\lambda} \in \Lambda(\mathcal{A})$. $\square$

Consequently, for $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$, we can modify the definition of $\varrho(\mathcal{A})$ to be

$$\varrho(\mathcal{A}) = \max\left\{|\lambda| \mid \lambda \in \mathcal{Z}(\mathcal{A})\right\}.$$

We hereby define the following two special values associated with $\Lambda(\mathcal{A})$:

$$\lambda^* := \max_{\lambda \in \Lambda(\mathcal{A})} \lambda \qquad \text{and} \qquad \lambda_* := \min_{\lambda \in \Lambda(\mathcal{A})} \lambda.$$

It is obvious $\lambda^* \leq \varrho(\mathcal{A})$. We shall investigate the possible circumstances for which $\lambda^* = \varrho(\mathcal{A})$. With this in mind, we next target the class of (weakly) symmetric tensors.

We first recall the following from [1,25].

Definition 3.8. $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called symmetric if

$$a_{i_1 \cdots i_m} = a_{\sigma(i_1 \cdots i_m)} \quad \text{for all } \sigma \in \mathfrak{S}_m,$$

where $\mathfrak{S}_m$ denotes the permutation group of m indices.

The notion of weakly symmetric tensors is introduced in [1].

Definition 3.9. $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is called weakly symmetric if the associated homogeneous polynomial

$$f_{\mathcal{A}}(x) := \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$$

satisfies $\nabla f_{\mathcal{A}}(x) = m\mathcal{A}x^{m-1}$. In the tensor notation, according to [25], the homogeneous polynomial $f_{\mathcal{A}}(x)$ is also denoted by $\mathcal{A}x^m$.

Although this definition is not as intuitive as symmetric tensors, it nevertheless provides the same desired variational (extremal) property as symmetric tensors. It should also be noted for $m = 2$, symmetric matrices and weakly symmetric matrices coincide. However, it is shown in [1] that a symmetric tensor is necessarily weakly symmetric for $m > 2$, but the converse is not true in general. Furthermore, if $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is weakly symmetric, by homogeneity, we find

$$\mathcal{A}x^m = f_{\mathcal{A}}(x) = \frac{1}{m}\langle \nabla f_{\mathcal{A}}(x), x\rangle = \langle \mathcal{A}x^{m-1}, x\rangle, \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

Proposition 3.10. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be weakly symmetric.*

(1) $\mathcal{Z}(\mathcal{A})$ consists precisely of all critical values of $f_{\mathcal{A}}(x) = \mathcal{A}x^m$ on S^{n-1}, hence it is nonempty.
(2) The cardinality of $\mathcal{Z}(\mathcal{A})$ is finite.

Proof. From Eq. (4), we see that $\lambda \in \mathcal{Z}(\mathcal{A})$ if and only if λ is a critical value of $f_{\mathcal{A}}(x)$ on the constraint manifold S^{n-1}. Furthermore, since $f_{\mathcal{A}}(x)$ is a continuous function defined on the compact set S^{n-1}, it must attain both its absolute maximum value and its absolute minimum value, hence $\mathcal{Z}(\mathcal{A}) \neq \emptyset$; this proves (1). To prove (2), we will show the set of E-eigenvalues of $\mathcal{A}$ is in fact finite. It is shown in [5] that the number of E-eigenvalues for a symmetric tensor is finite. The argument for weakly symmetric tensors is essentially identical, hence omitted. □

For simplicity, we define

$$\bar{\lambda} := \max_{x \in S^{n-1}} f_{\mathcal{A}}(x) = \max_{x \in S^{n-1}} \mathcal{A}x^m.$$

By specializing in the subclass of weakly symmetric nonnegative tensors, we arrive at another major result of this section.

Theorem 3.11. *Assume $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is weakly symmetric. Then*

$$\bar{\lambda} = \lambda^* = \varrho(\mathcal{A}).$$

Proof. It will suffice to establish the following chain of inequalities:

$$\bar{\lambda} \le \lambda^* \le \varrho(\mathcal{A}) \le \bar{\lambda}.$$

First we show $\bar{\lambda} \le \lambda^*$. From Proposition 3.10, we have $\bar{\lambda} \in \mathcal{Z}(\mathcal{A})$. Since $\mathcal{A}$ is weakly symmetric, using the Lagrange multipliers, we can find a maximizer, say $\bar{x}$, of the function $\mathcal{A}x^m$ on the smooth manifold S^{n-1}. We have the following equation

$$\nabla \mathcal{A}\bar{x}^m = m\mathcal{A}\bar{x}^{m-1} = m\lambda\bar{x}$$

for some multiplier λ. However, since $\langle \mathcal{A}\bar{x}^{m-1}, \bar{x}\rangle = \bar{\lambda}$, we see that $\lambda = \bar{\lambda}$. So $\mathcal{A}\bar{x}^{m-1} = \bar{\lambda}\bar{x}$, i.e. $(\bar{\lambda}, \bar{x})$ is a Z-eigenpair of $\mathcal{A}$. We now show $(\bar{\lambda}, |\bar{x}|)$ is also a Z-eigenpair of $\mathcal{A}$, where the n-vector

$|\bar{x}| := (|\bar{x}_1|, \ldots, |\bar{x}_n|)$ for $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$. Since $\bar{x} \in S^{n-1}$, by definition, $\mathcal{A}|\bar{x}|^m \leq \bar{\lambda}$. However, from the inequality:

$$\bar{\lambda} = \mathcal{A}\bar{x}^m = \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1\cdots i_m}\bar{x}_{i_1}\cdots\bar{x}_{i_m} \leq \sum_{i_1,i_2,\ldots,i_m=1}^{n} a_{i_1\cdots i_m}|\bar{x}_{i_1}|\cdots|\bar{x}_{i_m}| = \mathcal{A}|\bar{x}|^m,$$

it follows $|\bar{x}|$ is also a maximizer of $\mathcal{A}x^m$ on S^{n-1}. Using the Lagrange multipliers, it yields $\mathcal{A}|\bar{x}|^{m-1} = \bar{\lambda}|\bar{x}|$, i.e. $\bar{\lambda} \in \Lambda(\mathcal{A})$. Thus, $\bar{\lambda} \leq \lambda^*$.

Since $\Lambda(\mathcal{A}) \subset \mathcal{Z}(\mathcal{A})$, $\lambda^* \leq \varrho(\mathcal{A})$ holds.

Lastly, we show $\varrho(\mathcal{A}) \leq \bar{\lambda}$. Let $x_0 \in S^{n-1}$ be a corresponding Z-eigenvector of $\varrho(\mathcal{A})$. We then have:

$$\varrho(\mathcal{A}) = \langle \mathcal{A}x_0^{m-1}, x_0\rangle = \mathcal{A}x_0^m \leq \mathcal{A}|x_0|^m \leq \bar{\lambda};$$

hence, it completes the proof. □

Corollary 3.12. *Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be weakly symmetric and not equal to the zero tensor. Then $\lambda^* = \max_{x\in P\cap S^{n-1}} \mathcal{A}x^m > 0$.*

Proof. The fact that $\lambda^* = \max_{x\in P\cap S^{n-1}}$ follows immediately from Theorem 3.11. Suppose $\lambda^* = 0$. Since $\lambda^* = \bar{\lambda}$ and $\mathcal{A} \geq 0$, it follows that $\mathcal{A}x^m = 0$ for all $x \in P \cap S^{n-1}$. Hence $\mathcal{A}$ must be the zero tensor itself, a contradiction. □

4. A max–min characterization of λ^*

In this section, we establish a max–min property of λ^*, which is similar to half of the Collatz's type (see Theorem 2.4) of characterizations of the largest positive H-eigenvalue of $\mathcal{A}$. However, the other half of the statement fails as will be illustrated by a specific example at the end of this section.

In order to explore the max–min property of a nonnegative tensor, we assume tensors to be irreducible, as we did for H-eigenvalue problems.

Definition 4.1. Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$. We say $\mathcal{A}$ is non-degenerate if for all $x \in P \setminus \{0\}$, $(\mathcal{A}x^{m-1})_i$ and x_i do not vanish simultaneously for all $i \in \{1, \cdots, n\}$.

We now show if $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is non-degenerate. Suppose for some $\bar{x} \in P \setminus \{0\}$, $\bar{x}_{i_0} = 0$ and $(\mathcal{A}\bar{x}^{m-1})_{i_0} = \sum_{i_2,\ldots,i_m=1}^{n} a_{i_0 i_2\cdots i_m}\bar{x}_{i_2}\cdots\bar{x}_{i_n} = 0$ for some $i_0 \in \{1, \cdots, n\}$. Since each summand in $(\mathcal{A}\bar{x}^{m-1})_{i_0}$ is nonnegative and $\bar{x}_{i_k}$ cannot all be zero, this is only possible if $a_{i_0 i_2\cdots i_m} = 0$ for all $1 \leq i_2, \cdots, i_m \leq n$. In particular, $a_{i_0 i_2\cdots i_m} = 0$ for all $i_2, \cdots i_m \in \{1, \cdots, n\} \setminus \{i_0\}$, which implies $\mathcal{A}$ is reducible. The converse is not true in general, e.g. see example 4.6.

Definition 4.2. Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be non-degenerate. We define the following two functions for all $x \in P \setminus \{0\}$:

$$\nu_*(x) := \min_{1\leq i\leq n} \frac{(\mathcal{A}x^{m-1})_i}{x_i} \quad \text{and} \quad \nu^*(x) := \max_{1\leq i\leq n} \frac{(\mathcal{A}x^{m-1})_i}{x_i}.$$

Since $\mathcal{A}$ is non-degenerate, the fractions $\left\{\frac{(\mathcal{A}x^{m-1})_i}{x_i} | 1 \leq i \leq n\right\}$ are well defined over the extended reals for all $x \in P \setminus \{0\}$. Hence, $\nu_*(x)$ and $\nu^*(x)$ are both well defined. Note $\nu^*(x) = \infty$ can only happen on the boundary $\partial P \setminus \{0\}$. Since $\nu_*(x) \leq \nu^*(x)$ pointwise and $\nu_*(x) \leq \nu^*(x) < \infty$ for all $x \in P^\circ \cap S^{n-1}$, $\nu_*(x)$ and $\nu^*(x)$ are both continuous in $P^\circ \cap S^{n-1}$.

Definition 4.3. We define

$$\varrho_* := \sup_{x\in P^\circ\cap S^{n-1}} \nu_*(x) \quad \text{and} \quad \varrho^* := \inf_{x\in P^\circ\cap S^{n-1}} \nu^*(x).$$

Lemma 4.4. *Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be non-degenerate. There exist two vectors x_* and $x^* \in P \cap S^{n-1}$ such that $\varrho_* = \nu_*(x_*)$ and $\varrho^* = \nu^*(x^*)$, i.e.*

$$\max_{x\in P\cap S^{n-1}} \min_{1\le i\le n} \frac{(\mathcal{A}x^{m-1})_i}{x_i} = \varrho_*$$

$$\min_{x\in P\cap S^{n-1}} \max_{1\le i\le n} \frac{(\mathcal{A}x^{m-1})_i}{x_i} = \varrho^*.$$

Proof. By definition, the function ν_* is nonnegative and continuous on $P \cap S^{n-1}$. Therefore, $\max_{x\in P\cap S^{n-1}} \nu_*(x) = \sup_{x\in P^\circ\cap S^{n-1}} \nu_*(x) = \varrho_*$. Consequently, there exists $x_* \in P \cap S^{n-1}$ such that $\nu_*(x_*) = \max_{x\in P\cap S^{n-1}} \nu_*(x) = \varrho_*$.

Similarly, the function ν^* is nonnegative and lower semi-continuous on $P \setminus \{0\}$, which is bounded below by ϱ^*. By the compactness of $P \cap S^{n-1}$, there exists $x^* \in P \cap S^{n-1}$ such that $\nu^*(x^*) = \min_{x\in P\cap S^{n-1}} \nu^*(x) = \varrho^*$. □

We shall prove that ϱ^* and ϱ_* are the respective lower and upper bounds of $\Lambda(\mathcal{A})$ if $\mathcal{A}$ is irreducible, namely,

Theorem 4.5. *If $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is irreducible, then for all $\lambda \in \Lambda(\mathcal{A})$, we have: $0 < \varrho^* \le \lambda \le \varrho_*$, i.e. $\Lambda(\mathcal{A})$ is contained in the closed interval $[\varrho^*, \varrho_*]$.*

Proof. We first prove $\varrho^* > 0$ by contradiction. Suppose $\varrho^* = 0$. Then there exists $x_0 \in P\cap S^{n-1}$ such that $\nu^*(x_0) = 0$, which implies $\mathcal{A}x_0^{m-1} = 0$, i.e. x_0 is an eigenvector with eigenvalue 0. Since $\mathcal{A} \ge 0$ is irreducible, this contradicts Theorem 2.6. Let $\lambda \in \Lambda(\mathcal{A})$. Again by irreducibility, we see that $\lambda > 0$ and there exists $x \in P^\circ \cap S^{n-1}$ such that $\mathcal{A}x^{m-1} = \lambda x$. Hence, $\nu^*(x) = \lambda = \nu_*(x)$, which implies $\varrho^* \le \lambda \le \varrho_*$. □

In order to ensure $\Lambda(\mathcal{A}) \subseteq [\varrho^*, \varrho_*]$, the assumption of irreducibility on $\mathcal{A}$ is crucial. Otherwise, we have the following example.

Example 4.6. Let $\mathcal{A} \in \mathbb{R}_+^{[4,2]}$ be a symmetric tensor defined by

$$a_{1122} = \frac{2}{3} \quad \text{and} \quad a_{ijkl} = 0 \quad \text{elsewhere}.$$

Then the corresponding function $\mathcal{A}x^4 = 4x_1^2x_2^2$ on the quarter unit circle $P \cap S^1$ is equivalent, in polar form, to

$$\mathcal{A}x^4 = 4\cos^2\theta\sin^2\theta = \sin^2(2\theta) = \frac{1}{2} - \frac{1}{2}\cos(4\theta) \quad \text{for} \quad 0 \le \theta \le \pi/2.$$

This function achieves its absolute and local maximum at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ with its maximal value of 1; it also achieves its absolute but not local minimum at the points $(0, 1)$ and $(1, 0)$ with its minimal value of 0. It is easy to see that

1. $\lambda^* = 1$ with its corresponding Z-eigenvector $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$;
2. $\lambda_* = 0$ with its corresponding Z-eigenvectors $(0, 1)$ and $(1, 0)$;

3. $\nu_*(x) = \min\{2x_2^2, 2x_1^2\} = \begin{cases} 2x_2^2, & \text{for } 0 \le x_2 \le x_1 \\ 2x_1^2, & \text{for } x_1 \le x_2 \le 1; \end{cases}$

4. $\nu^*(x) = \max\{2x_2^2, 2x_1^2\} = \begin{cases} 2x_1^2, & \text{for } 0 \le x_2 \le x_1 \\ 2x_2^2, & \text{for } x_1 \le x_2 \le 1. \end{cases}$

Hence, $\varrho^* = \varrho_* = 1/2$; it is clear $\Lambda(\mathcal{A}) = \{0, 1\} \not\subseteq [\varrho^*, \varrho_*] = \{1/2\}$.

Remark. It is possible to have the strict inequality $\varrho^* < \varrho_*$; see Example 2.7.

If $\mathcal{A}$ is further assumed to be irreducible, we can extend the equalities obtained in Theorem 3.11 and obtain the following:

Theorem 4.7. *Assume $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is weakly symmetric and irreducible. Then $\varrho(\mathcal{A}) = \bar{\lambda} = \lambda^* = \varrho_*$.*

Proof. It follows from Theorem 4.5 $\varrho^* \le \lambda_* \le \lambda^* \le \varrho_*$. It remains to show $\lambda^* \ge \varrho_*$. By Lemma 4.4, there exists $x_0 \in P \cap S^{n-1}$ such that $\nu_*(x_0) \ge \varrho_*$. So we have:

$$\nu_*(x_0) = \min_{1 \le i \le n} \frac{(\mathcal{A}x_0^{m-1})_i}{(x_0)_i} \ge \varrho_*.$$

It follows that $(\mathcal{A}x_0^{m-1})_i \ge (\varrho_*)(x_0)_i$ for all $1 \le i \le n$. We thereby obtain:

$$\mathcal{A}x_0^m = \langle \mathcal{A}x_0^{m-1}, x_0 \rangle \ge \varrho_*.$$

Thus, we have $\lambda^* = \bar{\lambda} \ge \varrho_*$. This completes our proof. □

It is important to point out the equality $\lambda^* = \varrho_*$ may not hold in general if we drop the weakly symmetric assumption. We demonstrate this by revisiting Example 3.4. We consider the function

$$g_1(x) := \frac{(\mathcal{A}x^3)_1}{x_1} = 30x_1x_2 + x_2^2 + \frac{x_2^3}{x_1}$$

and the function

$$g_2(x) := \frac{(\mathcal{A}x^3)_2}{x_2} = 6\frac{x_1^3}{x_2} + 13x_1^2 + 37x_1x_2.$$

Changing to polar coordinates, we have for $0 \le \theta \le \pi/2$

$$g_1(\theta) = 30\cos\theta\sin\theta + \sin^2\theta + \frac{\sin^3\theta}{\cos\theta}$$

$$g_2(\theta) = 6\frac{\cos^3\theta}{\sin\theta} + 13\cos^2\theta + 37\cos\theta\sin\theta.$$

We find that $\varrho_* \approx 16.08381$ but the two curves intersect at $\lambda^* = 12.6$.

In a similar fashion, we can validate the calculation of the values ϱ^* and ϱ_* for Example 2.7. For simplicity, we set the function

$$g_1(x) := \frac{(\mathcal{A}x^3)_1}{x_1} = \frac{\frac{4}{\sqrt{3}}x_1^3 + 3x_1^2x_2 + x_2^3}{x_1} = \frac{4}{\sqrt{3}}x_1^2 + 3x_1x_2 + \frac{x_2^3}{x_1}$$

and the function

$$g_2(x) := \frac{(\mathcal{A}x^3)_2}{x_2} = \frac{x_1^3 + 3x_1x_2^2 + \frac{4}{\sqrt{3}}x_2^3}{x_2} = \frac{x_1^3}{x_2} + 3x_1x_2 + \frac{4}{\sqrt{3}}x_2^2.$$

Changing to polar coordinates, we have for $0 \le \theta \le \pi/2$

$$g_1(\theta) = \frac{4}{\sqrt{3}}\cos^2\theta + 3\cos\theta\sin\theta + \frac{\sin^3\theta}{\cos\theta}$$
$$g_2(\theta) = \frac{\cos^3\theta}{\sin\theta} + 3\cos\theta\sin\theta + \frac{4}{\sqrt{3}}\sin^2\theta$$

Therefore,

$$g_1(\theta) - g_2(\theta) = 2\cos(2\theta)\left[\frac{2}{\sqrt{3}} - \frac{1}{\sin(2\theta)}\right].$$

The two curves $g_1(\theta)$ and $g_2(\theta)$ intersect at three different points where $\theta = \frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$.

It is evident

$$\varrho^* = g_1\left(\frac{\pi}{4}\right) = g_2\left(\frac{\pi}{4}\right) = 2 + \frac{2}{\sqrt{3}}$$

and

$$\varrho_* = g_1\left(\frac{\pi}{6}\right) = g_2\left(\frac{\pi}{6}\right) = g_1\left(\frac{\pi}{3}\right) = g_2\left(\frac{\pi}{3}\right) = \frac{11}{2\sqrt{3}}.$$

From the discussions given above, one expects some kind of duality result between the pair (ϱ_*, λ^*) and the pair (ϱ^*, λ_*) for weakly symmetric nonnegative irreducible tensors. Unfortunately, we have an example to show the equality $\varrho^* = \lambda_*$ fails to hold in certain cases.

Example 4.8. Let $\mathcal{A} \in \mathbb{R}_+^{[4,2]}$ be a symmetric tensor defined by

$$a_{1111} = \frac{1}{2}, \quad a_{2222} = 3, \quad \text{and} \quad a_{ijkl} = \frac{1}{3} \quad \text{elsewhere.}$$

We then have:

$$(\mathcal{A}x^3)_1 = \frac{1}{2}x_1^3 + x_1^2x_2 + x_1x_2^2 + \frac{1}{3}x_2^3 \quad \text{and} \quad (\mathcal{A}x^3)_2 = \frac{1}{3}x_1^3 + x_1^2x_2 + x_1x_2^2 + 3x_2^3.$$

It follows that $\mathcal{A}$ is irreducible and the functions $g_1 := \frac{(\mathcal{A}x^3)_1}{x_1}$ and $g_2 := \frac{(\mathcal{A}x^3)_2}{x_2}$, under polar coordinates, are given by

$$g_1(\theta) = \frac{1}{2}x_1^2 + x_1x_2 + x_2^2 + \frac{1}{3}\frac{x_2^3}{x_1} = \frac{1}{2}\cos^2\theta + \frac{1}{2}\sin(2\theta) + \sin^2\theta + \frac{1}{3}\frac{\sin^3\theta}{\cos\theta}$$
$$g_2(\theta) = \frac{1}{3}\frac{x_1^3}{x_2} + x_1^2 + x_1x_2 + 3x_2^2 = \frac{1}{3}\frac{\cos^3\theta}{\sin\theta} + \cos^2\theta + \frac{1}{2}\sin(2\theta) + 3\sin^2\theta.$$

On the interval $\theta \in [0, \pi/2]$, the graph of $g_2(\theta)$ stays above the graph of $g_1(\theta)$ for $0 \le \theta \le \theta^* \approx 1.40687$. They have only one intersection at $\theta^* \approx 1.40687$, which corresponds to the value $\lambda^* = \varrho_* \approx 3.10921$ and the corresponding approximated Z-eigenvector is $(0.1632, 0.9866)$; thus

$$\Lambda(\mathcal{A}) = \{\lambda_* = \lambda^* \approx 3.10921\}.$$

However, since $\nu^*(x) = g_2(x)$ on the interval $[0, \theta^*]$ and $\nu^*(x) = g_1(x)$ on $[\theta^*, \pi/2]$, $\varrho^* = \min_{x \in P^\circ \cap S^1} \nu^*(x)$ is attained at $\theta_* \approx 0.42677184$ with $\varrho^* \approx 2.32694 < \lambda_*$.

When we compare Examples 2.7 and 4.8, we find the tensors in both examples are nonnegative symmetric and irreducible. However, in Example 2.7, $\varrho^* = \lambda_*$, whereas, in Example 4.8, $\varrho^* < \lambda_*$.

Inspired by Example 4.8, we define $\underline{\lambda} := \min_{x \in P \cap S^{n-1}} \mathcal{A}x^m$. Then $\underline{\lambda} \geq 0$. We have the following partial "dual" result to Theorems 3.11 and 4.7.

Proposition 4.9. *Let $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ be weakly symmetric and irreducible. Suppose the function $\mathcal{A}x^m$ attains its local minimum value $\underline{\lambda}$ in $P^\circ \cap S^{n-1}$, then*

$$\underline{\lambda} = \lambda_* = \varrho^*.$$

Proof. We proceed by showing $\lambda_* \leq \underline{\lambda} \leq \varrho^* \leq \lambda_*$. Suppose $\underline{\lambda}$ is attained at some interior point $x_* \in P^\circ \cap S^{n-1}$. Since $P^\circ \cap S^{n-1}$ is open in S^{n-1}, it is itself a differentiable manifold. The Lagrange multipliers therefore yields $\mathcal{A}x_*^{m-1} = \underline{\lambda}x_*$; hence, $\lambda_* \leq \underline{\lambda}$. Next we show $\underline{\lambda} \leq \varrho^*$. By Lemma 4.4, there exists $x_0 \in P \cap S^{n-1}$ such that $(\mathcal{A}x_0^{m-1})_i \leq \varrho^*(x_0)_i$ for all $1 \leq i \leq n$ where $(x_0)_i > 0$. So

$$\mathcal{A}x_0^m = \langle \mathcal{A}x_0^{m-1}, x_0 \rangle = \sum_{i=1}^{n} (\mathcal{A}x_0^{m-1})_i (x_0)_i \leq \varrho^* ||x_0||^2 = \varrho^*.$$

Since $\underline{\lambda} = \min_{x \in P \cap S^{n-1}} \mathcal{A}x^m$, we have $\underline{\lambda} \leq \varrho^*$. The last inequality $\varrho^* \leq \lambda_*$ follows from Theorem 4.7. □

Remark. The conclusion of Proposition 4.9 can be verified by Example 5.1 in the subsequent section.

We end this section with a more direct lower bound on $\varrho(\mathcal{A})$:

Corollary 4.10. *If $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$ is weakly symmetric, then $\varrho(\mathcal{A}) \geq \max\{c_1, c_2\}$, where*

$$c_1 = \max_{1 \leq i \leq n}\{a_{i \cdots i}\} \quad \text{and} \quad c_2 = \left(\frac{1}{\sqrt{n}}\right)^{m-2} \min_{1 \leq i \leq n} \sum_{i_2, \ldots, i_m = 1}^{n} a_{ii_2 \cdots i_m}.$$

Proof. On one hand, we choose $\{e_i \mid 1 \leq i \leq n\}$, the standard orthonormal basis for $\mathbb{R}^n$. Fix an index $i \in \{1, \ldots, n\}$, by Theorem 4.7, we have:

$$\varrho(\mathcal{A}) = \varrho_* \geq \frac{(\mathcal{A}(e_i)^{m-1})_i}{(e_i)_i} = a_{i \cdots i};$$

so $\varrho_* \geq c_1$. On the other hand, we choose $\mathbf{1} = (1, \cdots, 1)$, the vector whose entries are all ones. Then we have:

$$\varrho_* \geq \nu_* \left(\frac{\mathbf{1}}{\sqrt{n}}\right) = \min_{1 \leq i \leq n} \frac{\sum_{i_2, \ldots, i_m = 1}^{n} a_{ii_2 \cdots i_m} \left(\frac{1}{\sqrt{n}}\right)^{m-1}}{\frac{1}{\sqrt{n}}} = c_2. \quad \square$$

5. The algorithmic aspect

In this section, we adapt an iterative algorithm to compute λ^* when m is even, known as the Shifted Symmetric Higher-Order Power Method (SS-HOPM), proposed by Kolda and Mayo; see [13]. Although the algorithm as well as its convergence analysis are given under the assumption $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is symmetric, the entire process nonetheless continues to work successfully when we only assume $\mathcal{A}$ is weakly symmetric. Since the proof is identical, it will be omitted. We refer the interested reader to [13] for a more in-depth discussion on this subject.

We now adapt the SS-HOPM (cf. Algorithm 2 [13]) as follows. Given a weakly symmetric tensor $\mathcal{A} \in \mathbb{R}_+^{[m,n]}$.

Step 0. Choose $x_{(0)} \in P \setminus \{0\}$, set $\lambda_0 = \mathcal{A}x_{(0)}^m$, and choose the shift constant

$$\alpha = \lceil m \sum_{i_1,\cdots,i_m=1}^{n} a_{i_1\ldots i_m} \rceil,$$

where $\lceil \gamma \rceil$ is the ceiling function, i.e. it equals the smallest integer no less than γ. Set $k := 0$.

Step 1. Set $y_{(k+1)} := \mathcal{A}x_{(k)}^{m-1} + \alpha x_{(k)}$.

Step 2. Compute

$$x_{(k+1)} := \frac{y_{(k+1)}}{\|y_{(k+1)}\|}$$
$$\lambda_{k+1} := \mathcal{A}x_{(k+1)}^m.$$

The choice of α is far too conservative according to [13]. We now supply a brief outline of the main idea used in the proof. By choosing the shift constant $\alpha > 0$ large enough, the function $\hat{f}_{\mathcal{A}}(x) = \mathcal{A}x^m \pm \alpha\|x\|^m$ becomes convex or concave on $\mathbb{R}^n$. Hence when the order m is even, based on a result due to Kofidis and Regalia [14], the convergence of the sequence $\{\lambda_k\}$ is guaranteed.

Example 5.1. Consider the symmetric tensor $\mathcal{A} \in \mathbb{R}_+^{[4,2]}$ given by

$$a_{1111} = 1.1, \quad a_{2222} = 1.2, \quad a_{1112} = a_{1222} = \frac{1}{4}, \quad \text{and} \quad a_{ijkl} = 0 \quad \text{elsewhere}.$$

It defines

$$\mathcal{A}x^4 = 1.1x_1^4 + x_1^3x_2 + x_1x_2^3 + 1.2x_2^4,$$

hence we choose $\alpha = \lceil 4(1.1 + 2 + 1.2) \rceil = 18$. After we run 200 iterations using MATLAB, the above algorithm produces:

1. If we start with the initial point $(1, 1)$, then $\lambda_k \to 1.3040$.
2. If we start with the initial point $(0, 1)$, then $\lambda_k \to 1.3040$.
3. If we start with the initial point $(1, 0)$, then $\lambda_k \to 1.2139$.
4. For many other randomly chosen initial point $x_0 \in P^\circ$, the iterations will yield either the eigenvalue 1.3040 or 1.2139.

Both of these values are very good approximations. Since this example is of moderate size, we can compute its eigenpairs directly. We begin with

$$(\mathcal{A}x^3)_1 = 1.1x_1^3 + \frac{3}{4}x_1^2x_2 + \frac{1}{4}x_2^3$$
$$(\mathcal{A}x^3)_2 = \frac{1}{4}x_1^3 + \frac{3}{4}x_1x_2^2 + 1.2x_2^3.$$

The two curves $g_1(x) = \frac{(\mathcal{A}x^3)_1}{x_1}$ and $g_2(x) = \frac{(\mathcal{A}x^3)_2}{x_2}$ intersect at the following points (approximately):

(i) The point $(0.73386, 0.67929)$ corresponding to the eigenvalue

$$\underline{\lambda} = \varrho^* = \lambda_* \approx 1.07307.$$

(ii) The point $(0.971953, 0.235176)$ corresponding to the eigenvalue $\lambda_1 \approx 1.21394$.

(iii) The point $(0.212318, 0.977201)$ corresponding to the eigenvalue

$$\varrho_* = \lambda^* \approx 1.30396.$$

There is however an interesting aspect of the algorithm that deserves mentioning. Since $\lambda_* \approx 1.07307$ is a local minimum of $\mathcal{A}x^4$, we can also obtain this by reversing the sign of α, i.e. shifting by -18 instead. We verify the approach by starting with the initial point $(1, 1)$; after 200 iterations, the algorithm yields the approximated value $\lambda \approx 1.0731$.

We comment that although the exact number of solutions for symmetric tensors of this nature was also studied in Section 3.5 [7], the exact number of positive solutions was not deduced.

It is true in general, if λ_* is a local minimum of $\mathcal{A}x^m$ on $P^\circ \cap S^{n-1}$, then by reversing the sign of α in the above algorithm, one can also reach λ_*. Furthermore, since we are using a large enough α to guarantee the convexity of $\hat{f}_{\mathcal{A}}(x)$, the upper bound of $\varrho(\mathcal{A})$ is useful, but not the lower bound of $\varrho(\mathcal{A})$.

It is claimed by [13] that the SS-HOPM also works for m odd. Lastly, we demonstrate via a concrete example the SS-HOPM is not strictly limited to weakly symmetric tensors only.

Example 5.2. Let $\mathcal{A} \in \mathbb{R}_+^{[3,3]}$ be defined by

$$a_{122} = a_{133} = a_{211} = a_{311} = 1 \quad \text{and} \quad a_{ijk} = 0 \quad \text{elsewhere.}$$

The eigenvalue problem is to solve:

$$\begin{cases} x_2^2 + x_3^2 = \lambda x_1, \\ x_1^2 = \lambda x_2, \\ x_1^2 = \lambda x_3, \\ x_1^2 + x_2^2 + x_3^2 = 1. \end{cases}$$

We choose $\alpha = 12$ with initial point $(1, 1, 1)$ (among many other choices). After we run 100 iterations using MATLAB, the above algorithm produces $\lambda_k \to 0.8381$ with the corresponding $\hat{x} \approx \pm(0.6652, 0.5280, 0.5280)$.

On the other hand, using the computational commutative algebra system CoCoA[6], we consider the ideal generated by

$$\{x_2^2 + x_3^2 - \lambda x_1,\ x_1^2 - \lambda x_2,\ x_1^2 - \lambda x_3,\ x_1^2 + x_2^2 + x_3^2 - 1\}.$$

By computing the elimination ideal via eliminating the variables x_1, x_2, and x_3, we come up with the E-characteristic polynomial $\psi_{\mathcal{A}}$ of $\mathcal{A}$:

$$\psi_{\mathcal{A}}(\lambda) = 3\lambda^6 + 6\lambda^4 - 4;$$

whose only two real zeros are $\lambda^* \approx 0.8381016549$ and -0.8381016549.

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A. Appendix

- Proof of Theorem 2.5. As for the H-eigenvalue problem, we reduce the algebraic problem to a fixed point problem. Let $D = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, 1 \leq i \leq n, \sum_{i=1}^n x_i = 1\}$ be the unit simplex, then D is a closed convex set. Suppose there exists some $u_0 \in D$ such that $\mathcal{A}u_0^{m-1} = 0$. Let $\lambda_0 = 0$

and $x_0 = u_0/||u_0||$, then (λ_0, x_0) is a solution to equation (2), and we are done. Thus, one may assume $\mathcal{A}x^{m-1} \neq 0$ for all $x \in D$; the following map $F : D \to D$ is therefore well defined:

$$F(x)_i = \frac{(\mathcal{A}x^{m-1})_i}{\sum_{j=1}^n (\mathcal{A}x^{m-1})_j}, \quad 1 \le i \le n,$$

where $(\mathcal{A}x^{m-1})_i$ represents the i-th component of $\mathcal{A}x^{m-1}$. The map $F : D \to D$ is clearly continuous. According to the Brouwer's Fixed Point Theorem, there exists a $v_0 \in D$ such that $F(v_0) = v_0$, i.e. $\mathcal{A}v_0^{m-1} = \tau v_0$ for some $\tau = \sum_{j=1}^n (\mathcal{A}v_0^{m-1})_j \ge 0$. Finally, we normalize v_0 to produce the nonnegative Z-eigenvector $x_0 = v_0/||v_0||$ with the normalized Z-eigenvalue $\lambda_0 = \tau/||v_0||^{m-2}$ as required.

- Proof of Theorem 2.6. We first prove $x_0 \in P^\circ$, i.e. assertion (2). Note $P \setminus P^\circ = \partial P = \cup_{I \in \Pi} F_I$, where Π is the set of all index subsets I of $\{1, \ldots, n\}$ and

$$F_I = \{(x_1, \ldots, x_n) \in P \mid x_i = 0 \ \forall i \in I \ \text{and} \ x_j \neq 0 \ \forall j \notin I\}.$$

Suppose $x_0 \notin P^\circ$. Since $x_0 \neq 0$, there must be a maximal proper index subset $I \in \Pi$ such that $x_0 \in F_I$, i.e. $(x_0)_i = 0 \ \forall i \in I$ and $(x_0)_j > 0 \ \forall j \notin I$. Let $\delta = \min\{(x_0)_j \mid j \notin I\}$, we then have $\delta > 0$. Since x_0 is an eigenvector, $\mathcal{A}x_0^{m-1} \in F_I$, i.e.

$$\sum_{i_2,\ldots,i_m=1}^{n} a_{ii_2\cdots i_m}(x_0)_{i_2}\cdots(x_0)_{i_m} = 0, \quad \forall i \in I.$$

It follows that

$$\delta^{m-1} \sum_{i_2,\ldots,i_m \notin I} a_{ii_2\cdots i_m} \le \sum_{i_2,\ldots,i_m \notin I} a_{ii_2\cdots i_m}(x_0)_{i_2}\cdots(x_0)_{i_m} = 0, \quad \forall i \in I,$$

hence we have $a_{ii_2\cdots i_m} = 0$ for all $i \in I$ and for all $i_2, \ldots, i_m \notin I$; according to Lemma 2.2 [2], $\mathcal{A}$ is reducible, which is a contradiction.

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