April 18, 2019

Last time: network centrality

1. PageRank ☆

$$\alpha P x + (1-\alpha) v = x$$

2. HITS (Hubs & Auth)

$$A = U \Sigma V^T$$

auth $= v_1$, hub $= u_1$

3. Eigenvector ☆

$$A x = \lambda_1 x$$

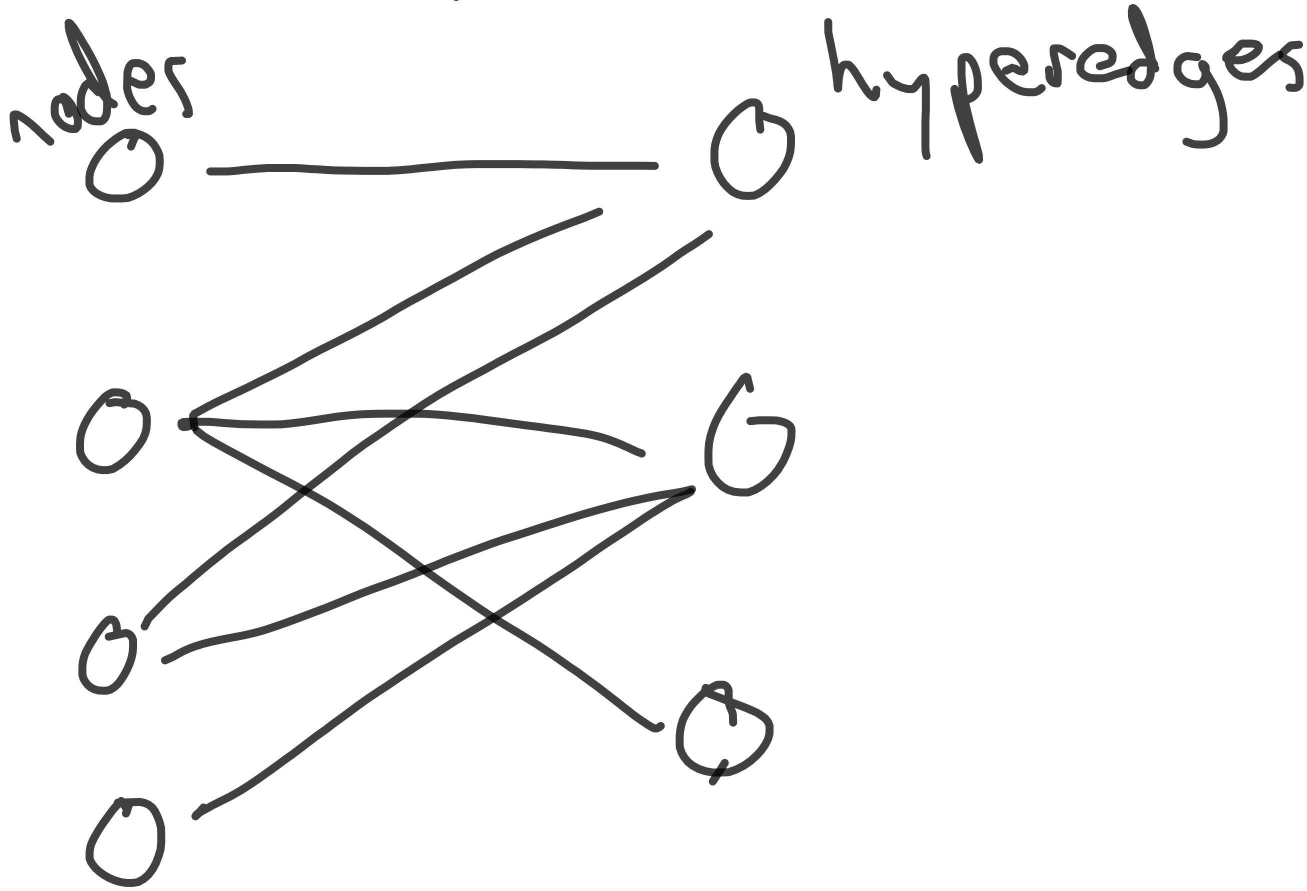
4. Katz

$$(I - \beta A^T) k = \beta A^T \mathbf{1}$$

Hypergraphs

instances of small sets

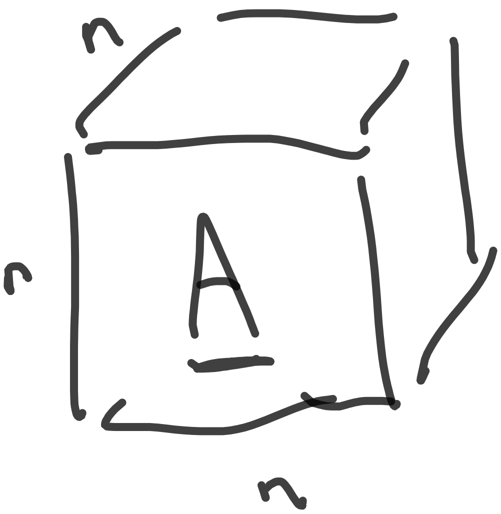
- multiple recipients on email
- authors on a paper
- multiple drugs/medications for patients



Adjacency tensor

simple model:

- size-3 relationships
- no direction

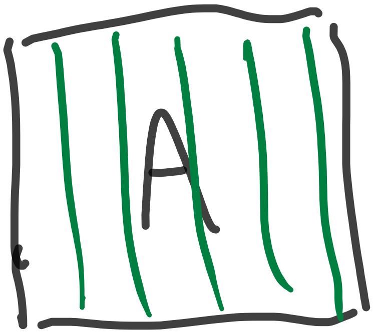


$$\underline{A}_{ijk} = \begin{cases} 1 & \text{if } \{i,j,k\} \text{ is in dataset} \\ 0 & \text{o/w} \end{cases}$$

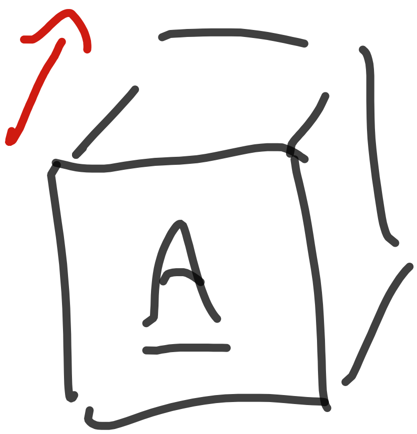
Tensor-based PageRank (Gleich, Lim, Yu 2015)

$$\alpha P x + (1-\alpha) v = x$$

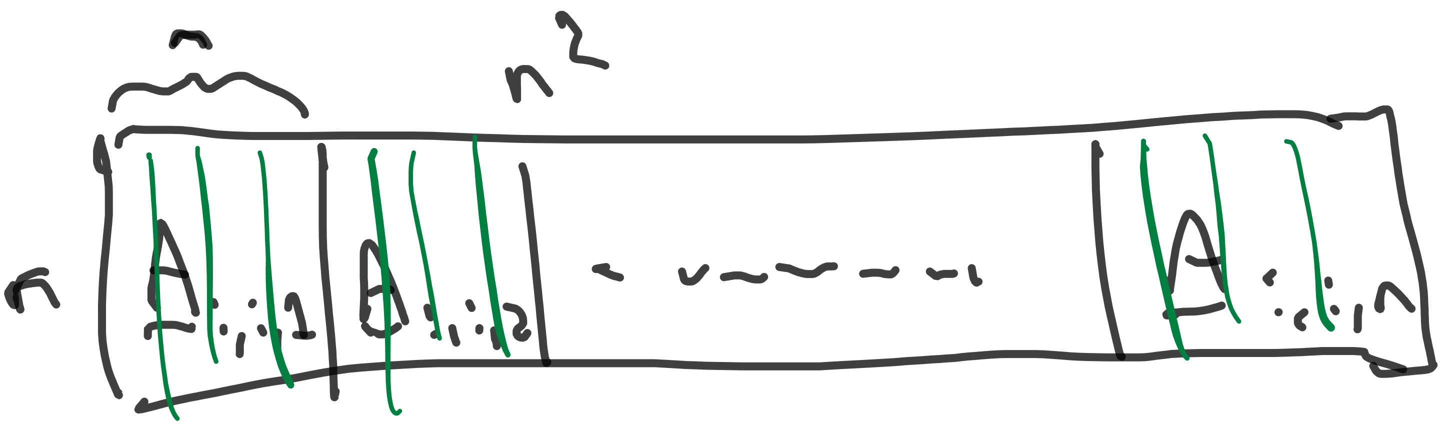
What is "P" matrix for tensors?



columns normalized to sum to 1

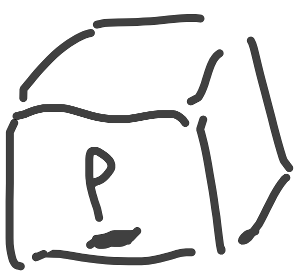


"unfold"



normalize to sum to 1

refolding



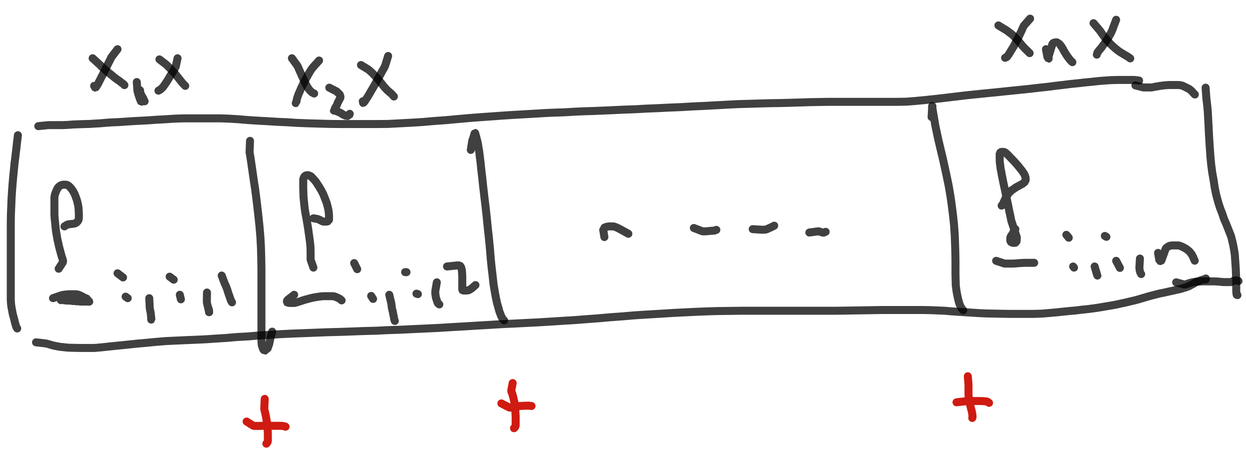
Matrix case

$$\alpha P x + (1-\alpha) v = x$$

Tensor case

$$\boxed{\alpha \underline{P} x^2 + (1-\alpha) v = x}$$

$$[\underline{P} x^2]_i = \sum_{jk} \underline{P}_{ijk} x_j x_k$$



$$= \underline{P} x^2$$

Does a solution even exist?

Yes!

Claim: $f(x) = \alpha \underline{P} x^2 + (1-\alpha) v$

is a stochastic vector

$(\underline{1}^T f(x) = 1 \quad f(x) \geq 0)$

Proof: $\underline{P} x^2 =$

$$\begin{array}{cccc} x_1 x & x_2 x & & x_n x \\ \hline \multicolumn{1}{|c|}{\underline{P}_{:,:,1}} & \multicolumn{1}{c|}{\underline{P}_{:,:,2}} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{\underline{P}_{:,:,n}} \\ \hline \end{array}$$

$\underline{P}_{:,j,k}$ is stochastic

$y_j = \underline{P}_{:,:,j} x$ is stochastic

$\underline{P} x^2 = \sum x_j y_j$ is stochastic

$f(x) = \alpha \underline{P} x^2 + (1-\alpha) v$ is stochastic

$$\alpha P x^2 + (1-\alpha)\cancel{x} = x$$

$\parallel$

$f(x)$ is stochastic as long as x is stochastic

Theorem (Brouwer fixed point):

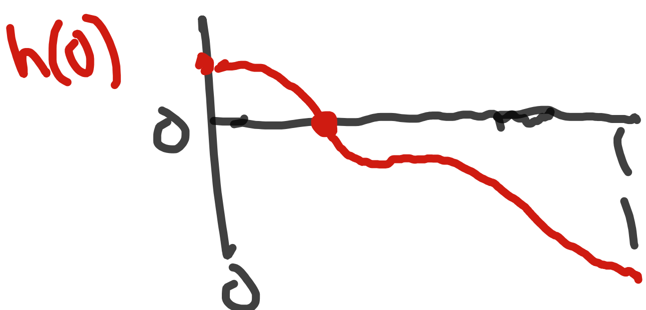
Let $g: K \to K$ be a smooth function on a compact convex set K. Then there exists $x \in K$ such that $g(x) = x$

Proof: for $K = [0,1]$

$h(x) = g(x) - x$

$\exists\, y$ s.t. $h(y) = 0$

$\Leftrightarrow g(y) = y$



$$K = \{ w \in \mathbb{R}^n \mid 1^T w = 1, w \geq 0 \}$$

$$f : K \to K$$

$$f(x) = \alpha \underline{P} x^2 + (1-\alpha) v$$

BFP $\Rightarrow \exists\, x$ such that

$$\alpha \underline{P} x^2 + (1-\alpha) v = x$$

Can we actually compute a solution?

Matrix

$$x_{k+1} = \alpha P x_k + (1-\alpha) v$$

Tensor

☆ $$x_{k+1} = f(x_k) = \alpha \underline{P} x_k^2 + (1-\alpha) v$$

Theorem (Gleich, Lim, Yu)

if $\alpha < 1/2$

- ☆ converges (again, exponentially)
- solution $\alpha \underline{P} x^2 + (1-\alpha)v = x$ is unique

if $\alpha \geq 1/2$

- ☆ doesn't necessarily converge
- solution may not be unique

Hypergraph eigenvector centrality (Benson 2019)

graph evec centrality

$$Ax = \lambda_1 x \qquad x > 0$$

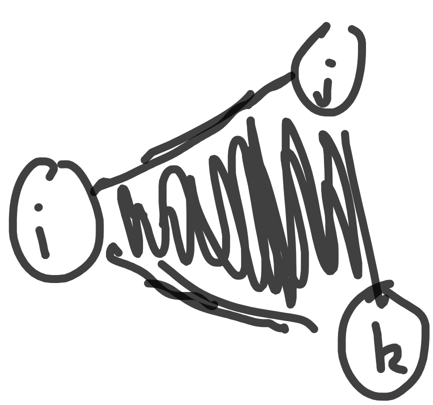
power

$$x_i = \gamma \sum_{(i,j)\in E} x_j \quad \leftarrow \text{power of } j$$

$$\lambda_1 = \frac{1}{\gamma}$$

theoretical tool:

Perron-Frobenius theorem



power of node i prop. to powers of all such j, k

① $$x_i = \gamma \sum_{(i,j,k)} x_j x_k$$

$$\Rightarrow \underline{A} x^2 = \frac{1}{\gamma} x$$

z-eigenvector of a tensor

② $$x_i^2 = \gamma \sum_{(i,j,k)} x_j x_k$$

$$\Rightarrow \underline{A} x^2 = \frac{1}{\gamma} x^2$$

$$[x^2]_i = x_i^2$$

H-eigenvector of $\underline{A}$

(3) $x_i = \gamma \sum_{(i,j,k)} x_j + x_k$

$\Rightarrow Wx = \frac{1}{2\gamma} x$

$W_{ij} = \#\, k$ such that

$\underline{A}_{ijk} = 1$

$W = \sum_k A_{:,:,k}$

Can apply P-F like last time

$\Rightarrow$ unique $x > 0$ such that

$Wx = \lambda_1 x$

$\lambda_1 > 0$

$$\left.\begin{array}{ll}(Z) & \underline{A}\, x^2 = \lambda x \\ (H) & \underline{A}\, x^2 = \lambda x^2\end{array}\right\}$$ also want $x > 0$

Start with H

Definition: $\underline{A}$ is <u>irreducible</u> if W ($W_{i} = \sum \underline{A}_{ijk}$) induces a connected graph

Theorem: If $\underline{A}$ is irreducible and nonnegative, then exists unique $\lambda > 0$, $x > 0$ such that

$$\underline{A}\, x^2 = \lambda x^2$$

any other eigenvalue λ' has $|\lambda'| \leq \lambda$

$\Rightarrow$ this gives us the centrality vector

Can we compute it?

Yes! (Ng, Qi, Zhou 2009)

$$y_k = \underline{\underline{A}}\, x_k^2$$

$$x_{k+1} = \frac{y_k}{\| y_k^{1/2} \|_2}$$

entry-wise $\sqrt{\ }$

Z-eigenvector centrality

- Want:

$$\underline{A}\, x^2 = \lambda x$$

$$\lambda > 0, \; x > 0$$

Theorem: if $\underline{A}$ is irreducible, then there exists $\lambda > 0$, $x > 0$ such that $\underline{A}\, x^2 = \lambda x$

but....

- NP-hard to compute
- could be another $\lambda' > 0$, $y > 0$ such $\underline{A}\, y^2 = \lambda' y$

if we can compute some

$\lambda \geq 0$, $x \geq 0$ such that

$$\underline{A} x^2 = \lambda x$$

then we have tensor

Z-eigenvector centrality vector

Options:

- worst-case exponential time algorithms
- heuristics that tend to work very well in practice