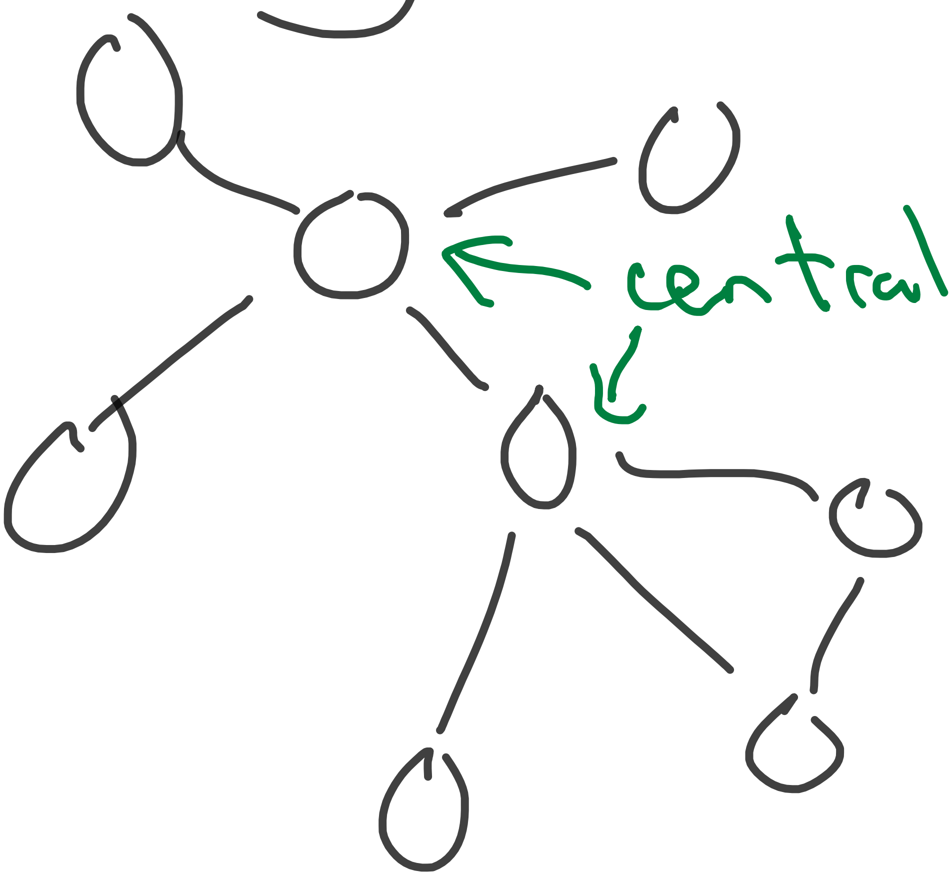
April 11, 2019

Centrality in networks

ranking



- PageRank
- HITS

} web

- Eigenvector
- Katz

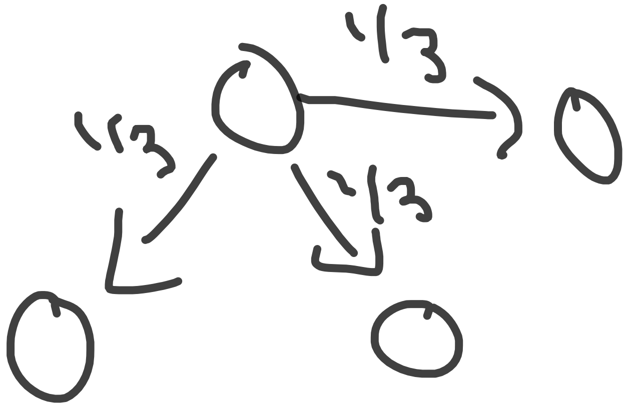
} sociology

PageRank (Page et al. 98)

(Assume directed, strongly connected graph)

Stochastic process:

- start at some node
- With prob α, follow an outgoing edge at random



- With prob. $1-\alpha$, "teleport" to a random node

telep. vector v

this is a Markov chain

steady state? (stationary dist)

$$\alpha P x + (1-\alpha) v = x$$

$P_{ji} = \mathrm{Prob}(i \to j)$

Since $\mathbb{1}^T x = 1$ at steady state,

$$\alpha P x + (1-\alpha) v \mathbb{1}^T x = x$$

$$\underbrace{(\alpha P + (1-\alpha) v \mathbb{1}^T)}_{\text{Markov chain matrix}} x = x$$

Alternatively, linear system

$$(I - \alpha P)x = (1-\alpha)v$$

How to compute x?

$$x = \alpha P x + (1-\alpha)v$$

$$x^{(k+1)} = \alpha P x^{(k)} + (1-\alpha)v$$

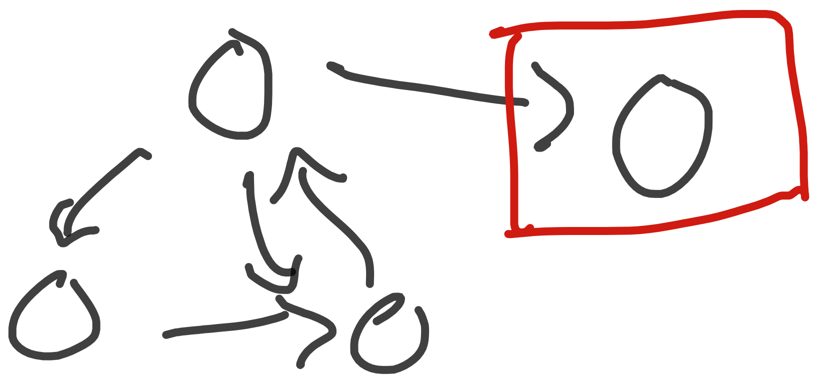
$$x - x^{(k+1)} = \alpha P(x - x^{(k)})$$

$$\begin{aligned}
\|x - x^{(k+1)}\|_1 &= \|\alpha P(x - x^{(k)})\|_1 \\
&\leq \alpha \|P\|_1 \|x - x^{(k)}\|_1 \\
&= \alpha \|x - x^{(k)}\|_1 \\
&\leq \alpha^k (\|x - x^{(0)}\|_1) \\
&\leq 2\alpha^k
\end{aligned}$$

exponential decay in error!

Problem when not strongly conn?

"sink nodes"



nowhere to go

if we land at one of these, could just follow v

Theorem: equivalent to rescaling iterates at each step

$$x^{(k+1)} = \alpha P x^{(k)} + (1-\alpha) v$$

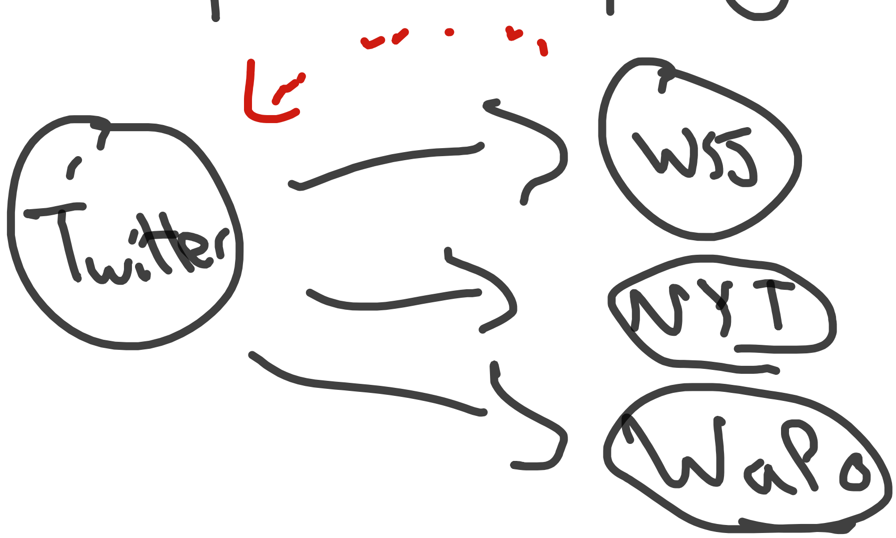
$$x^{(k+1)} = x^{(k+1)} / \|x^{(k)}\|_1$$

HITS (J. Kleinberg 98)
hubs & authorities

Main idea: web sites have "hub" and "authority" scores

hubs point to many good authorities

authorities are pointed to by many good hubs



h_0, a_0 initial hub and auth

$$h_1(i) = \sum_{(i,j)} a_0(j)$$

$$h_1 = A a_0$$

$$a_1(j) = \sum_{(i,j)} h_1(i) \qquad a_1 = A^T h_1$$

$$h_2 = A a_1 = A A^T h_1$$

$$= A A^T A a_0$$

$$a_2 = A^T h_2 = A^T A A^T A a_0$$

$$= (A^T A)^2 a_0$$

$$h_k = (A A^T)^{k-1} \cdot A a_0$$

$$a_k = (A^T A)^k a_0$$

$$h_k \leftarrow \frac{h_k}{\|h_k\|_2}$$

$$a_k \leftarrow \frac{a_k}{\|a_k\|_2}$$

Running power method on AA^T (h_k) or A^TA (a_k)

$\Rightarrow$ converge to the principal eigenvector (assuming its unique)

$A = U \Sigma V^T$

$AA^T = U \Sigma^2 U^T$ (eigendecomp)

$\Rightarrow$ h_k converges to u_1

(first left singular vector)

$$A^T A = V \Sigma^2 V^T$$

$\Rightarrow a_k$ converges to v_1

(first right singular vector)

We could have initialized a_0 as nonnegative

u_1, v_1 are both nonnegative (can be made nonnegative)

(uniqueness of singular vectors up to sign)

Eigenvector centrality
(Bonacich 1972)

Assume node i has some "power" or "influence" v_i

Power is proportional to sum of your neighbors' power

$$v_i = \gamma \sum_{(i,j)} v_j$$

proportionality constant > 0

$$\Leftrightarrow \frac{1}{\gamma} v_i = \sum_{j=1}^{n} A_{ij} v_j$$

$$= [Av]_i$$

$$\Leftrightarrow \frac{1}{\gamma} v = A v$$

Eigenvalue problem!

This could hold for any eigenpair $(\frac{1}{\gamma}, v)$, $\frac{1}{\gamma} > 0$

Assume $v > 0$

if graph is strongly conn., then we can find such an eigenpair and its unique

Theorem (Perron-Frobenius)

Let A be the adj. matrix of a strongly conn. graph

Then there exists $v > 0$ such that

1. $Av = \lambda v$
2. $\lambda > 0$ is an eigenvalue of largest magnitude of A
3. the eigenspace of λ is one-dimensional
4. v is the only nonneg.

eigenvector of A
(up to scaling)

1 + 2 $\Rightarrow$ which eigenvector
(largest real eigenval)

3 + 4 $\Rightarrow$ uniqueness

if the graph is not
k-partite, then $|\hat{\lambda}| < d$
$\hat{\lambda}$ is any other eigenvalue

$\Rightarrow$ can use simple power
method to compute

Katz centrality (Katz 1953)

node i is important if there are many paths to node i

shorter paths more important than longer ones

Katz: weight paths of length l by β^l

Katz score of node i

$$K_i = \sum_{l=1}^{\infty} \sum_{j=1, j \neq i}^{n} \beta^l \cdot (\text{\# paths from } j \text{ to } i \text{ of}$$

length ℓ)

$$A^{\ell}_{ji}$$

$$K_i = \sum_{\ell=1}^{\infty} \sum_{j=1} A^{\ell}_{ji}$$

$$(I - M)^{-1} = \sum_{\ell=0}^{\infty} M^{\ell} \quad (M^0 = I)$$

$$K = [(I - \beta A^T)^{-1} - I] \mathbb{1}$$

need β small enough to conv.

$$\beta < \|A^T\|_2^{-1} = \sigma_1^{-1}$$

$$\Rightarrow \|\beta A\|_2 < 1$$