Feb 21, 2019

Today:

1. More on eigenvectors for tensors

   Lim '05

   Qi '05

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2. dimensionality redux with tensors

Matrix evecs

$$Ax = \lambda x$$

$$\sum A_{ij} x_j = \lambda x_i \qquad i = 1, \ldots, n$$

opt. problem:

A symm.

$$\max_x \; x^T A x = \sum A_{ij} x_i x_j$$

$$\text{s.t.} \quad \|x\|_2^2 = 1$$

$$x^T x = 1$$

$$x^T x - 1 = 0$$

$$\mathcal{L}(x, \lambda) = x^T A x - \lambda (x^T x - 1)$$

$\nabla_{x,\lambda} \mathcal{L} = 0 \Leftarrow$ at any local optima

$$\nabla_x \mathcal{L} = 2Ax - \lambda 2x$$

$$= 0 \iff Ax = \lambda x$$

local optima are eigenvectors with eigenvalues are Lagrange multipliers

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Tensor analog $\underline{A}$ symm.

$$\max_x \sum \underline{A}_{ijk} x_i x_j x_k$$

$$\text{s.t.} \quad \|x\|_2^2 = 1$$

$$\mathcal{L}(x, \lambda) = \sum \underline{A}_{ijk} x_i x_j x_k - \lambda(x^T x - 1)$$

$$\nabla_x \mathcal{L} = \left[\sum_{jk} 3 \underline{A}_{ijk} x_j x_k\right]_i - 2\lambda x$$

$$\stackrel{\Delta}{=} 3\underline{A}x^2 - 2\lambda x$$

$$= 0 \iff \underline{A}x^2 = \left(\tfrac{2}{3}\lambda\right)x$$

"Z eigenvector" ~~local optima~~ extreme point

Matrix case scale-invariant

$$Ax = \lambda x$$

$$A(\alpha x) = \alpha(Ax) = \alpha\lambda x = \lambda(\alpha x)$$

Z-evecs not scale invar.

$$\underline{A}(\alpha x)^2 = \alpha^2 \underline{A}x^2 = \alpha^2 \lambda x$$

$$= \alpha\lambda(\alpha x)$$

"require" $\|x\|_2 = 1$ for well-defined Z-eigenvalues order-4: $\underline{A}x^3 = \lambda x$

alternative

$$\max_x \sum \underline{A}_{ijk} x_i x_j x_k$$

$$\text{s.t.} \quad \|x\|_3^2 = 1$$

$$\|x\|_3^3 = \sum |x_i|^3$$

$$\mathcal{L}(x, \lambda) = \sum \underline{A}_{ijk} x_i x_j x_k - \lambda\left(\|x\|_3^3 - 1\right)$$

$$\nabla_x \mathcal{L} = 3\underline{A}x^2 - 3\lambda\left(\frac{x \circ x}{\|x\|_3}\right)$$

$$\left((y \circ z)_i = y_i z_i\right)$$

$$= 0 \iff \underline{A}x^2 = \tilde{\lambda}\, x \circ x$$

Scale-invariance

$$\underline{A}\,(\alpha x)^2 = \alpha^2\,\underline{A}\,x^2$$
$$= \alpha^2\,\lambda\, x \circ x$$
$$= \lambda\,(\alpha x) \circ (\alpha x)$$

"H eigenvectors"

Dimensionality reduction with tensor data

Matrix

$$A \approx U_k\,\Sigma_k\,V_k^T \qquad U_k^T U_k = V_k^T V_k = I$$

$$= \sum_{r=1}^{k} \sigma_r\, u_r v_r^T$$

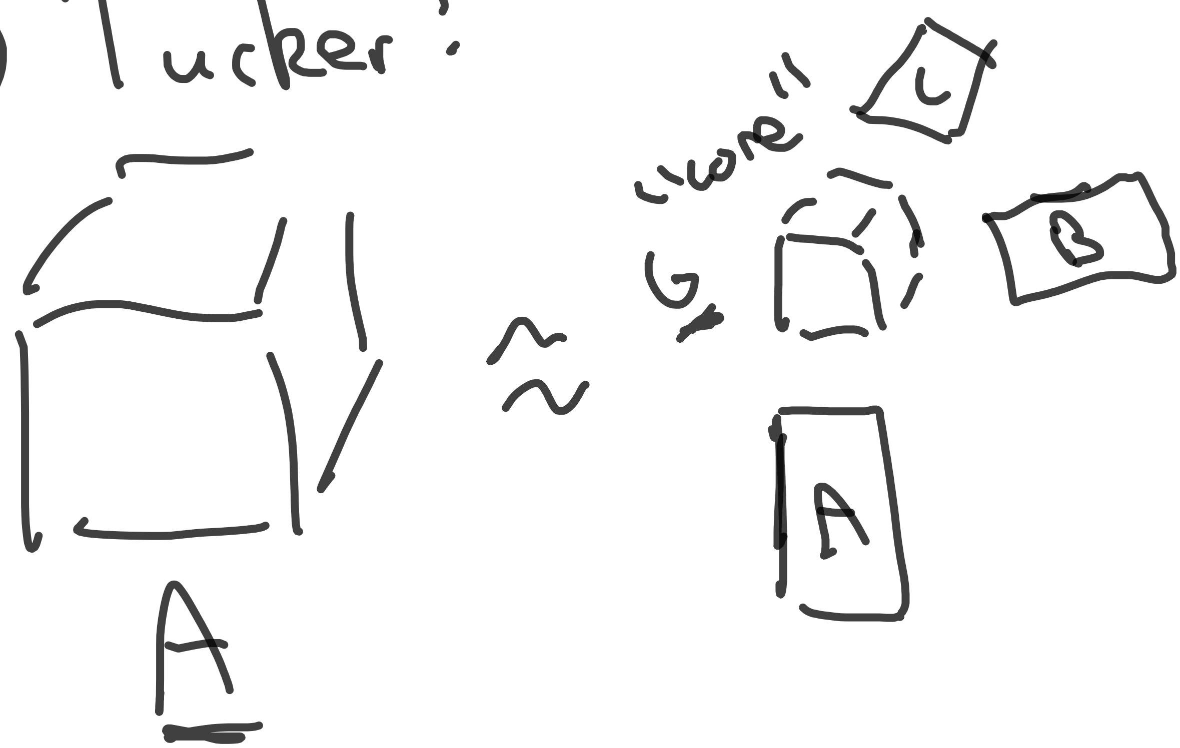
$$= \sum_{r=1}^{k} \sigma_r u_r \otimes v_r$$

best rank-k

Two types of tensor decomps

① CP: $$\underline{A} \approx \sum_{r=1}^{k} \sigma_r a_r \otimes b_r \otimes c_r$$

② Tucker:



Canonical decomposition

CANDECOMP

Parallel Factors

PARAFAC

$\boxed{\text{CP}}$

Canonical polyadic

How do we compute?

$$\underline{A} \approx \sum (\sigma_r \, a_r) \otimes b_r \otimes c_r$$

optional

$$\min_{a,b,c} \left\| \underline{A} - \sum a_r \otimes b_r \otimes c_r \right\|_F$$

$$\left(\underline{A}_{ijk} - \sum a_r(i)\, b_r(j)\, c_r(k)\right)^2$$

$$\underbrace{\begin{bmatrix} & \vdots & \\ b_1(j)c_1(k) & \cdots & b_r(j)c_r(k) \\ & \vdots & \end{bmatrix}_{j,k}}_{M}$$

$$\left(M \begin{bmatrix} a_1(i) \\ \vdots \\ a_r(i) \end{bmatrix} - \cancel{\approx} \begin{bmatrix} \vdots \\ \underline{A}_{ijk} \\ \vdots \end{bmatrix} \right)^2$$

linear least squares

Actual alg:

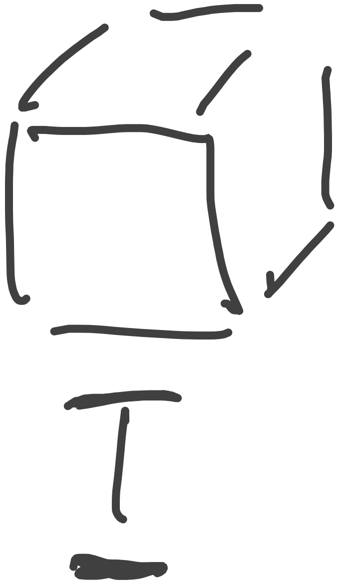
for iter = 1, 2, ...

Fix B, C
Update A with LLS
Fix A, C
Update B with LLS

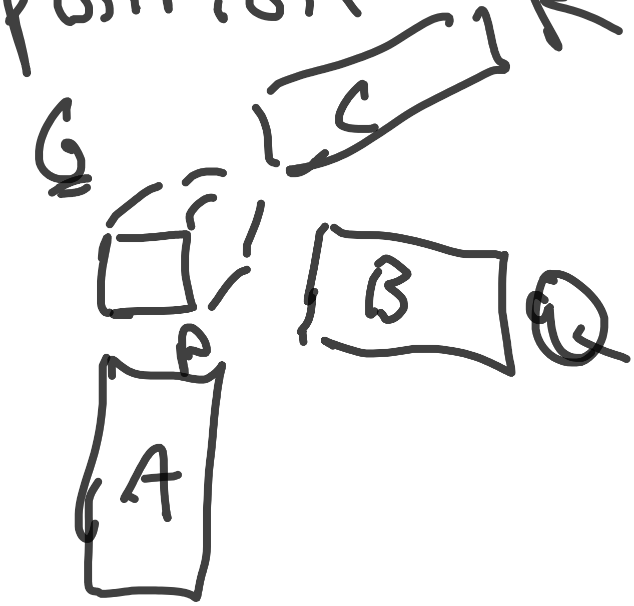
Fix A, B
Update C with LLS

stop if error < ε

Tucker decomposition



$\approx$



$\underline{G}$ is the "coretensor"

$$\underline{T} \approx \sum_{p=1}^{P} \sum_{q=1}^{Q} \sum_{r=1}^{R} \underline{G}_{pqr}\, a_p \otimes b_q \otimes c_r$$

$A^T A = I$

$B^T B = I$

$C^T C = I$

sometimes called "higher-order SVD"

$\underline{G}$ diagonal
+ no orthog. constraints
$\Longrightarrow$ CP

How to compute?

Idea: fix all but one factor
$\Longrightarrow$ matrix M
$\Longrightarrow$ SVD $U\Sigma V^T$
$\Longrightarrow$ update M with U

$$\underline{T} \approx \sum_p \sum_q \sum_r \underline{G}_{pqr}\, a_p \otimes b_q \otimes c_r$$

$$\underline{T}_{ijk} \approx \sum_{p,q,r} \underline{G}_{pqr}\, a_p(i)\, b_q(j)\, c_r(k)$$

Claim: if we know A, B, C are fixed, we can find optimal $\underline{G}$

$$\left\| \underline{T} - \sum_{pqr} \underline{G}_{pqr}\, a_p \otimes b_q \otimes c_r \right\|_F^2$$

$$\left(\underline{T}_{ijk} \approx \sum_{p,q,r} \underline{G}_{pqr}\, a_p(i)\, b_q(j)\, c_r(k)\right)^2$$

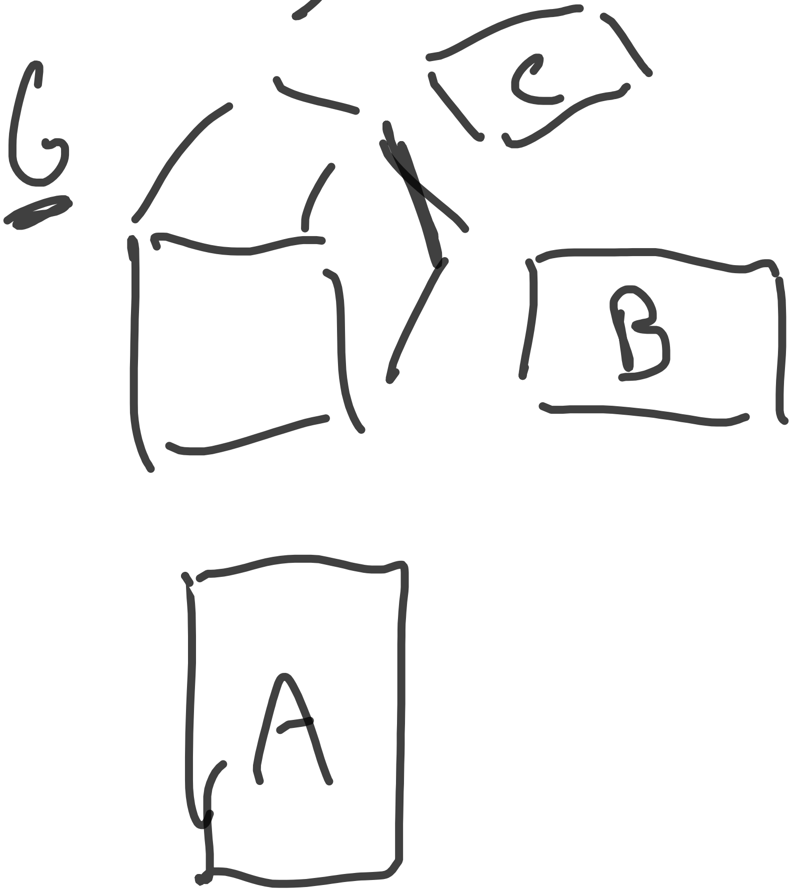
Solve with LLS

How to get A, B, C

Idea: capture variation in

each mode

$\Rightarrow$ left singular vectors



Collapse to a matrix in each mode (A, B, C)

$$\underline{T}_{ijk} \approx \sum_{p,q,r} \underline{G}_{pqr}\, a_p(i)\, b_q(j)\, c_r(k)$$

$$[X(A)]^T_{:,i} = A_{:,i} \sum_{jk} \sum_{qr} \underline{T}_{ijk}\, b_q(j)\, c_r(k)$$

(Kolda & Bader '09)

$$X(A) = U \Sigma V^T$$

$$n \; \overbrace{\boxed{U}}^{?}$$

$$A \leftarrow U$$

for iter = 1, 2, ...

$$\mathrm{SVD}(X(A)) \rightarrow U \Sigma V^T$$

$$A \leftarrow U$$

$$\mathrm{SVD}(X(B)) \rightarrow U \Sigma V^T$$

$$B \leftarrow U$$

$$\mathrm{SVD}(X(C)) \rightarrow U \Sigma V^T$$

$$C \leftarrow U$$

stop if error $\lesssim \varepsilon$

$\rightarrow$ LLS for $\underline{G}$

Generalizations

- nonnegative tensor factorizations

- I showed order-3 tensors generalizes for any order

Next Tues: no class
(break)

Next Thurs:
nonlinear dimensionality
redux