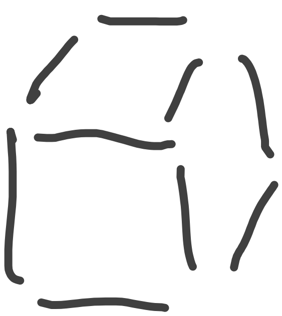
Feb 19, 2019

This week:

tensors + low-rank decompositions

today: issues/problems

Thurs: algs



$$\underline{A}_{ijk} = \begin{cases} 1 & \text{patient } i \text{ on prescription } j \text{ in week } k \\ 0 & \text{o/w} \end{cases}$$

Matrices

rank of A is the smallest R s.t. $A = \sum_{r=1}^{R} a_r b_r^T$

$= a_r \otimes b_r$

Tensors

rank of $\underline{A}$ is the smallest R s.t. $\underline{A} = \sum_{r=1}^{R} a_r \otimes b_r \otimes c_r$

$$\underline{A}_{ijk} = \sum_{r=1}^{R} a_r[i]\, b_r[j]\, c_r[k]$$

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best rank-k approx

$$X_k = \underset{X}{\arg\min} \|A - X\|_F$$

$$\text{s.t. } \operatorname{rank}(X) = k$$

$$X_k = \sum_{r=1}^{k} \sigma_r u_r \otimes v_r$$

$$= U_k \Sigma_k V_k^T$$

$$U_k^T U_k = V_k^T V_k = I$$

$$u_r^T u_s = v_r^T v_s = \delta_{rs}$$

$$X_{k+1} = X_k + \sigma_{k+1} u_{k+1} \otimes v_{k+1}$$

Eckart-Young-Mirsky

Tensors

sum of squares error

$$\underline{X}_k = \min_{\underline{X}} \| \underline{A} - \underline{X} \|_F^2$$

$$\text{s.t. } \operatorname{rank}(\underline{X}) = k$$

Problem 1 (Kolda '03)

No EYM theorem for tensors

$$\underline{A} = \sigma_1 u \otimes u \otimes u + \sigma_2 v \otimes v \otimes w$$

choose: $u^T w = 0$

$$v = \frac{1}{\sqrt{2}}(u + w)$$

$$\|u\|_2 = \|w\|_2 = 1$$

$$\sigma_1 > \sigma_2 > 0$$

$$\text{rank}(\underline{A}) = 2$$

Theorem: best rank-1 approx to $\underline{A}$ is $\sigma_1 x \otimes y \otimes z$, where

$$\|x\| = \|y\| = \|z\| = 1$$

$$a^T u, a^T v, a^T w \neq 0$$

$a = x, y,$ or z

$\Rightarrow$ no nesting necessary in best rank-k tensor approx.

Theorem (Kolda '01)
orthog. tensor decomp may not be unique

Theorem (Kolda '03):
No rank-2 decomposition of $\underline{A}$ where $*$ is one of the factors

Tensors

sum of squares error

OP: $$\underline{X}_k = \min_{\underline{X}} \|\underline{A} - \underline{X}\|_F^2$$
$$\text{s.t. } \operatorname{rank}(\underline{X}) = k$$

Problem 2 (de Silva - Lim '08)

OP can be ill-posed in the sense that there is no best rank-k approx to $\underline{A}$

$$\underline{A}_n = n\left(x_1 + \tfrac{1}{n} y_1\right) \otimes \left(x_2 + \tfrac{1}{n} y_2\right) \otimes \left(x_3 + \tfrac{1}{n} y_3\right) - n\, x_1 \otimes x_2 \otimes x_3$$

rank 2

$$\lim_{n\to\infty} \underline{A}_n = \underline{A}^* \quad \text{rank } 3$$

$$= x_1 \otimes x_2 \otimes y_3 + x_1 \otimes y_2 \otimes y_3 + y_1 \otimes x_2 \otimes x_3$$

Consequence: no best rank-2 approx to $\underline{A}^*$

Theorem: $d_1, d_2, d_3 \geq 2$

$2 \leq s \leq \max\{d_1, d_2, d_3\}$.

Then exists $\underline{A}$ with no best rank-r approx, $r < s$

<u>AND</u>

$\{\underline{A} \in \mathbb{R}^{d_1 \times d_2 \times d_3} \mid \underline{A}$ with no best rank-r approx$\}$

has positive volume

Problem 3: (Håstad '90)

In general, its hard to compute the rank of a tensor (NP-hard)

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① What about a symmetric tensor? $\underline{A}_{ijk} = \underline{A}_{ikj}$

$$= \underline{A}_{jik} = \underline{A}_{jki}$$

$$= \underline{A}_{kij} = \underline{A}_{kji}$$

② What about just best rank-1 approx?

Problem 4 ([Lim-Hillar '13])

Theorem: best symm. rank-1 approx. to symmetric tensors is NP-hard

$$\min_{\gamma, x} \|\underline{A} - \gamma\, x \otimes x \otimes x\|_F^2$$

s.t. $\|x\|_2 = 1$

$$= \sum_{ijk} \left(\underline{A}_{ijk} - \gamma x_i x_j x_k\right)^2$$

$$= \sum_{ijk} \cancel{\underline{A}_{ijk}^2} - 2\gamma \underline{A}_{ijk} x_i x_j x_k + \gamma^2 (x_i x_j x_k)^2$$

$$\Rightarrow \sum_{ijk} -2\gamma \underline{A}_{ijk} x_i x_j x_k + \gamma^2 (x_i x_j x_k)^2$$

$$\gamma^2 \sum_i \sum_j \sum_k x_i^2 x_j^2 x_k^2$$

$$\gamma^2 \underbrace{\cancel{\sum_i x_i^2}}_{1} \underbrace{\cancel{\sum_j x_j^2}}_{1} \underbrace{\cancel{\sum_k x_k^2}}_{1}$$

$$\gamma^2 - 2\gamma \sum_{ijk} \underline{A}_{ijk} x_i x_j x_k$$

$$\partial / \partial \gamma = 0 \iff$$

$$\sum_{ijk} \underline{A}_{ijk} x_i x_j x_k = \gamma$$

$$\hookrightarrow \gamma^2 - 2\gamma^2 = -\gamma^2$$

we would like to max.

over γ

⋮

$$\max \quad \gamma$$

tensor e-vec (Lim '05)

$$\text{s.t.} \quad \sum_{ijk} A_{ijk} x_i x_j x_k = \gamma$$

$$\|x\|_2 = 1$$

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matrix case

$$\max \quad \lambda$$

$x^T A x$

$$\text{s.t.} \quad \sum_{ij} A_{ij} x_i x_j = \lambda$$

$$\|x\| = 1$$

* Alt. form

$$\sum_{jk} \underline{A}_{ijk} x_j x_k = \lambda x_i$$

$$i = 1, \dots, n$$

$$\|x\|_2 = 1$$

Matrix case

$$\sum_{j} A_{ij} x_j = \lambda x_i$$

$$i = 1, \dots, n$$

$$\|x\|_2 = 1$$

Best sym. rank-1 approx to a tensor is a tensor eigenvector of largest eigenval

Notes:

1. still NP-hard to compute in general
2. best rank-2 not nec. given by first 2 tensor evecs
3. What about non-symm. approx?

$$\underset{\gamma, x, y, z}{\operatorname{argmin}} \ (*) \ \|\underline{A} - \gamma\, x \otimes y \otimes z\|_F^2$$

s.t. $\|x\| = \|y\| = \|z\| = 1$

Theorem (Banach '30)

$$\circledast = \underset{\gamma, x}{\arg\min} \; \| \underline{A} - \gamma \, x \otimes x \otimes x \|_F^2$$

$$\text{s.t.} \quad \|x\| = 1$$

Today: everything was hard

Next time: algs for low-rank approx