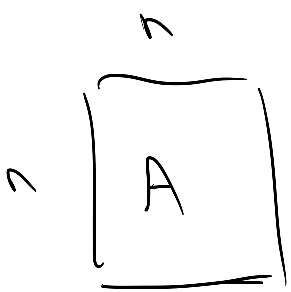
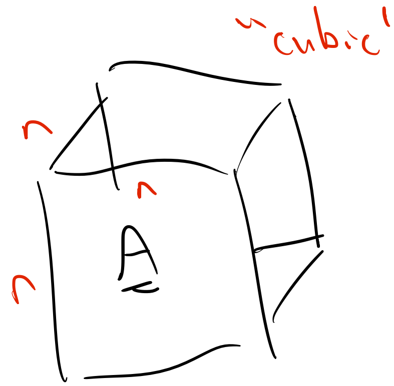
Dec 14, 2020 (GVL 12.4-12.5 4/e)

Tensors / hypermatrices are generalizations of matrices

a_i A_{ij} $\underline{A}_{ijk}$

$\underline{A}_{ijk}$ = info j of patient i at time k





$$A_{ij} = \begin{cases} 1 & \text{if } i,j \text{ contact} \\ 0 & \text{o/w} \end{cases}$$

$$\underline{A}_{ijk} = \begin{cases} 1 & i,j,k \text{ contact} \\ 0 & \text{o/w} \end{cases}$$

Rank and low-rank approx

SVD $\Rightarrow$ best low-rank approx

$$B_k = \underset{B}{\arg\min} \|A - B\|_F^2 \quad \text{s.t. } \operatorname{rank}(B) = k$$

$$A = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T$$

$$B_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

$$\operatorname{rank}(B) = \min r \quad \text{s.t. } B = \sum_{s=1}^{r} x_s y_s^T$$

$$B_{ij} = \sum_{s=1}^{r} (x_s)_i (y_s)_j$$

$$B = {}_{n}\overset{r}{\boxed{X}}\ {}_{r}\overset{n}{\boxed{Y^T}}$$

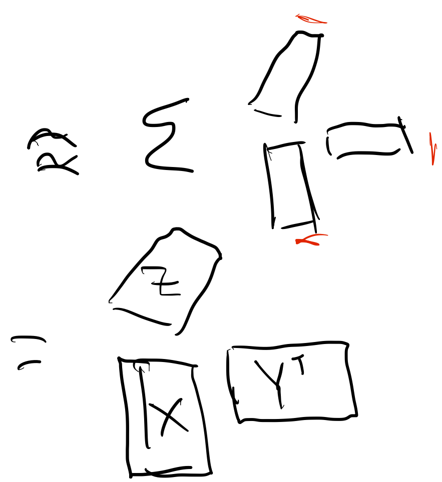
$$\underline{B}_k = \underset{\underline{B}}{\arg\min} \|\underline{A} - \underline{B}\|_F^2 \quad \text{s.t. } \operatorname{rank}(\underline{B}) = k$$

$$\|\underline{M}\|_F^2 = \sum_{ijk} |\underline{M}_{ijk}|^2$$

$$\operatorname{rank}(\underline{B}) = \min r \quad \text{s.t. } \underline{B} = \sum_{s=1}^{r} x_s \otimes y_s \otimes z_s$$

$$\underline{B}_{ijk} = \sum_{s=1}^{r} (x_s)_i (y_s)_j (z_s)_k$$

$\underline{B} \approx \sum$



Problems:
- computing $\mathrm{rank}(\underline{T})$ is NP-hard
- no easy way to determine max possible rank
- no nesting: $B_{k+1} = B_k + \sigma_{k+1} u_{k+1} v_{k+1}^T$ (matrix)
- ill-posed

$$\left.\begin{aligned} \underline{A}_n &= n\left(x_1 + \tfrac{1}{n} y_1\right) \otimes \left(x_2 + \tfrac{1}{n} y_2\right) \otimes \left(x_3 + \tfrac{1}{n} y_3\right) \\ &\quad - n\left(x_1 \otimes x_2 \otimes x_3\right) \end{aligned}\right\} \text{rank-2}$$

$$\underline{A}^* = \lim_{n \to \infty} \underline{A}_n \quad \left.\begin{aligned} & x_1 \otimes x_2 \otimes y_3 \\ &+ x_1 \otimes y_2 \otimes x_3 \\ &+ y_1 \otimes x_2 \otimes x_3 \end{aligned}\right\} \text{rank-3}$$

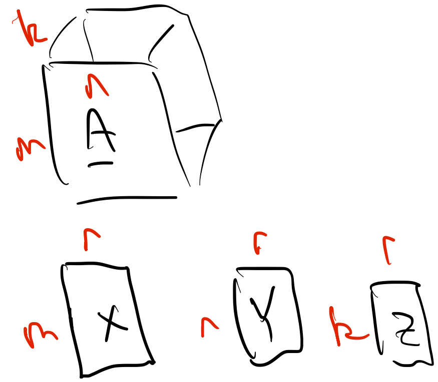
$$\cancel{+ n x_1 \otimes y_2 \otimes x_3 - n x_1 \otimes x_2 \otimes x_3}$$

$$\left.\begin{aligned} \min_{\underline{B}} \; & \|\underline{A}^* - \underline{B}\|_F \le \varepsilon \\ \text{s.t.} \; & \mathrm{rank}(\underline{B}) = 2 \end{aligned}\right\} \text{no solution}$$

border rank = 2

CP decomposition

$$\min_{X,Y,Z} \left\| \underline{A} - \sum_{s=1}^{r} x_s \otimes y_s \otimes z_s \right\|_F^2$$



$$\min_{X} \left\| \underline{A} - \sum_{s=1}^{r} x_s \otimes y_s \otimes z_s \right\|_F^2$$

$$= \sum_{ijk} \left(\underline{A}_{ijk} - \sum_{s=1}^{r} x_{is} y_{js} z_{ks} \right)^2$$

$$\underline{A}_{ijk} - \left[y_{j1} z_{k1} \cdots\cdots y_{jr} z_{kr} \right] \begin{pmatrix} x_{i1} \\ \vdots \\ x_{ir} \end{pmatrix}$$

$X(i,:)$

$$\left\| \left(\underline{A}_{ijk} \right)_{jk} - \left[y_{j1} z_{k1} \cdots\cdots y_{jr} z_{kr} \right]_{jk} X(i,:) \right\|_2^2$$

Khatri-Rao product $Z \odot Y$

$$\left\| \mathrm{vec}(A(i,:,:)) - (Z \odot Y)\, X(i,:) \right\|_2^2 \Rightarrow \left\| \underline{A}_{(1)} - (Z \odot Y) X^T \right\|_F^2$$

$$+ \lambda \left\| X(i,:) \right\|_2^2$$

ALS

(1) $$X_{k+1} = \arg\min_{X} \| A_{(1)} - (Z_k \odot Y_k) X^T \|_F^2 + \lambda \|X\|_F^2$$

(2) $$Y_{k+1} = \arg\min_{Y} \| A_{(2)} - (Z_k \odot X_{k+1}) Y^T \|_F^2 + \lambda \|Y\|_F^2$$

(3) $$Z_{k+1} = \arg\min_{Z} \| A_{(3)} - (Y_{k+1} \odot X_{k+1}) Z^T \|_F^2 + \lambda \|Z\|_F^2$$

- stop if error $\leq \varepsilon$
- regularization