Dec 11, 2020

HW7 due Wed Dec 16

OH: Austin Mon 2:45-3:45pm
John Tues 10-11am

Final:
- 5-7 questions
- some coding
- work on your own
- open everything (cite sources)

Released: Wed Dec 16
Due: Mon Dec 21 5pm ET (CMS)

Iterative methods for eigenproblems

$$\underset{x}{\text{minimize}} \quad \text{error}(Ax, b) \qquad \text{CG: } \|Ax-b\|_{A^{-1}} \iff \|x - A^{-1}b\|_A \quad Ax-b \perp \mathcal{K}_k$$
$$\text{s.t.} \quad x \in \mathcal{K}_k(A,b) \qquad \text{MINRES/GMRES/LSMR} \quad \|Ax-b\|_2$$

$$\underset{x,\lambda}{\text{minimize}} \quad \underline{\text{error}(A, x, \lambda)}$$
$$\text{s.t.} \quad x \in \mathcal{K}_k(A,b)$$

b starting/seed vector

Why might $\mathcal{K}_k(A,b) = \text{span}\{b, \dots, A^{k-1}b\}$

1. power method: $x_{k-1} = A^{k-1}b / \|A^{k-1}b\| \in \mathcal{K}_k$
2. CG/MINRES/GMRES: error analysis: $\displaystyle\min_{q_k \in P_k} \max_{\lambda_i} |q_k(\lambda_i)|^2 \qquad q(0)=1$
3. $\mathcal{K}_k(A - \sigma I, b) = \mathcal{K}_k(A,b)$
4. $Q_k^T A Q_k = T_k / H_k$ first step: $U^T A U = T / H$
5. only needed $n \boxed{X}_k \to \boxed{U} \to AX$ (orthog iter, power method)

⑥ $r(x) = x^T A x / x^T x \quad \nabla r(x^*) = 0 \Rightarrow A x^* = \lambda x^* \quad \lambda = r(x^*)$

$\lambda_1 = \max_{x \neq 0} r(x) \qquad \lambda_n = \min_{x \neq 0} r(x)$

$V_k = [v_1 \cdots v_k] \qquad V_k^T V_k = I$

$a_k = \lambda_1 (V_k^T A V_k) = \max_{0 \neq y} \dfrac{y^T V_k^T A V_k y}{y^T y} = \max_{z \in \text{span}(V_k)} r(z) \leq \lambda_1$

$= y^T V_k^T V_k y$

$b_k = \lambda_k (V_k^T A V_k) = \min_{z \in \text{span}(V_k)} r(z) \geq \lambda_n$

$a_1 \leq a_2 \cdots \leq a_n = \lambda_1 \quad b_1 \geq \cdots \geq b_n = \lambda_n$

Let $u_k = V_k y$ satisfying $r(u_k) = a_k$

For any grad-based opt. next step in $\nabla r(u_k) \in \text{span}(V_{k+1})$

$\nabla r(u_k) = \dfrac{2}{u_k^T u_k} (A u_k - r(u_k) u_k) \in \text{span}(u_k, A u_k)$

$\text{span}(v_1, \ldots, v_k) = \text{span}(v_1, A v_1, A^2 v_1, \ldots, A^{k-1} v_1)$

$= K_k(A, v_1)$

minimize instead maximize $\quad w_k = V_k y \quad \nabla r(w_k) \in \text{span}(V_{k+1})$

$(\hat{\lambda}, \hat{v})$ $\quad r = (A - \hat{\lambda} I)\hat{v} \perp K_k$

$\hat{v} \in K_k$

$\hat{v} = Q_k y$

$0 = Q_k^T (A - \hat{\lambda} I) Q_k y = (T_k - \hat{\lambda} I) y \Rightarrow \underline{T}_k y = \hat{\lambda} y$

cond holds for any eigenpair of T_k

k small $\Rightarrow$ evecs of T_k with direct algos

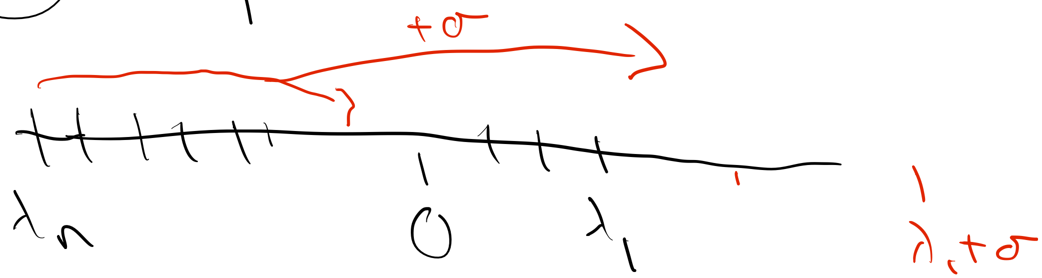
$A \neq A^T$

$0 = Q_k^T (A - \hat{\lambda} I) Q_k y = (H_k - \hat{\lambda} I) y \Rightarrow H_k y = \hat{\lambda} y$

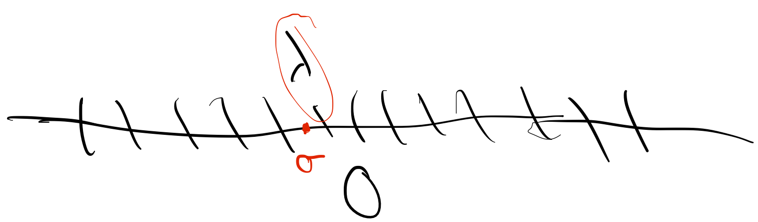
<u>Ritz</u> vectors and values

With this strategy...

1. At least as good as power method (largest magnitude)
2. Any shift $A - \sigma I$



extremal eigenpairs



smallest magnitude?

$(A - \sigma I)^{-1}$ has evals $\dfrac{1}{\lambda_i - \sigma}$

Search in $K_k((A - \sigma I)^{-1}, b)$

$$(A - \sigma I)^{-1} q_k \Rightarrow (A - \sigma I) z = q_k$$

several shifts at a time

What about the SVD? eigs / svds

$$A V_k = U_{k+1} \bar{\bar{B}}_k \qquad \text{span}(V_k) = K_k(A^T A, A^T b)$$

$$A^T U_k = V_{k+1} B_{k+1}^T$$

$$A = U \Sigma V^T \qquad A^T A = V \Sigma^2 V^T$$

$$\hat{u}, \hat{v}, \hat{\sigma} \qquad A\hat{v} \approx \hat{\sigma}\hat{u}$$

$$\hat{v} \in \text{span}(V_k) \qquad \hat{u} \in \text{span}(U_k)$$

$$0 = U_k^T (A\hat{v} - \hat{\sigma}\hat{u}) \qquad \hat{v} = V_k y \qquad \hat{u} = U_k z$$

$$0 = U_k^T A V_k y - \hat{\sigma} z,$$

$$B_k y = \hat{\sigma} z$$