Dec 7, 2020

$Ax = b \quad K_k(A, b) = \{b, Ab, \dots, A^{k-1}b\}$

	opt problem	subproblem	
MINRES $A = A^T$	$\min \|Ax - b\|_2$ s.t. $x \in K_k$	$y_k = \arg\min \|\bar{T}_k y - \beta_0 e_1\|_2$, $x_k = Q_k y_k$	Lanczos: $AQ_k = Q_{k+1}\bar{T}_k$, $\bar{T}_k = \begin{pmatrix} \alpha_1 & \beta_1 & \cdots & \\ \beta_1 & \ddots & & \beta_{k-1} \\ & \cdots & \beta_{k-1} & \alpha_k \\ & & & \beta_k \end{pmatrix}$
GMRES	$\min \|Ax - b\|_2$ s.t. $x \in K_k$	$y_k = \arg\min \|\bar{H}_k y - \beta_0 e_1\|_2$, $x_k = Q_k y_k$	$AQ_k = Q_{k+1}\bar{H}_k$, $q_1 = b/\|b\|$, $\bar{H}_k$ is $(k+1) \times k$

$x = Q_k y$

$\|AQ_k y - b\|_2$

$= \|Q_{k+1}\bar{H}_k y - b\|$ (Q_k^T)

$= \left\| \begin{pmatrix} I \\ 0 \end{pmatrix} \bar{H}_k y - \begin{pmatrix} \beta_0 \\ 0 \end{pmatrix} \right\|$

$= \|\bar{H}_k y - \beta_0 e_1\|$

$\bar{H}_k = G_k \begin{pmatrix} R_k \\ 0 \end{pmatrix}$ (G_k is $(k+1) \times (k+1)$, R_k is $k \times k$)

$G_k^T G_k = I$

$\|G_k^T \bar{H}_k y - G_k^T \beta_0 e_1\|_2$

$= \left\| \begin{pmatrix} R_k \\ 0 \end{pmatrix} y - \begin{pmatrix} z_k \\ \bar{\eta}_{k+1} \end{pmatrix} \right\|_2$

$R_k y_k = z_k$

residual $= \bar{\eta}_{k+1}$

$\bar{H}_{k+1} = \begin{pmatrix} \bar{H}_k & h_{k+1} \\ 0 & h_{k+1,k+1} \end{pmatrix}$

$\tilde{G}_k^T \cdots \tilde{G}_1^T \bar{H}_k = \begin{pmatrix} R_k \\ 0 \end{pmatrix}$

$\tilde{G}_k^T \cdots \tilde{G}_1^T \beta_0 e_1 = \begin{pmatrix} z_k \\ \bar{\eta}_{k+1} \end{pmatrix}$

$\tilde{G}_{k+1}^T \tilde{G}_k^T \cdots \tilde{G}_1^T \bar{H}_{k+1} = \begin{pmatrix} R_k & x \\ 0 & x \end{pmatrix}$

$\tilde{G}_{k+1}^T \begin{pmatrix} z_k \\ \bar{\eta}_{k+1} \\ 0 \end{pmatrix} \to \begin{pmatrix} z_{k+1} \\ \bar{\eta}_{k+2} \end{pmatrix}$

Problem: no short recurrence

Partial solution: restarted GMRES

1. Run m steps of GMRES $\to x_m$
2. $r = b - Ax_m$
3. Run m steps of GMRES

   $$Az = r \Rightarrow z_m$$
4. $x_m = x_m + z_m$
5. Repeat 2, 3, 4

$$A(x_m + z_m) = b - r + Az_m$$

Error analysis

$$\min \|Ax - b\| \quad \text{s.t. } x \in K_k$$

$$\min \|A\,p(A)\,b - b\|_2 \quad \text{s.t. } p \in P_{k-1}$$

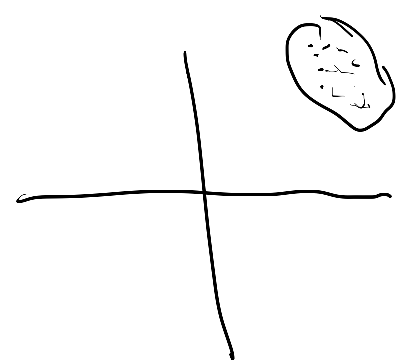
$$A = V\Lambda V^{-1}$$

$$\|V\Lambda \cancel{V^{-1}V}\, p(\Lambda) V^{-1} b - V V^{-1} b\|_2$$

$$\le \|V\| \, \|(I - \Lambda p(\Lambda)) V^{-1} b\|_2$$

$$\le \|V\| \, \|V^{-1}\| \, \|b\| \, \|q(\Lambda)\|$$

$$= \kappa_2(V) \, \|r_0\| \min_{\substack{q \in P_k \\ \text{s.t. } q(0) = 1}} \max_{\lambda_i} |q(\lambda_i)|$$



$$q(z) = (1 - z/c)^k$$

$$q(0) = 1$$

Choose c so that λ_i/c is small for all i

dependence $\kappa_2(A)$, $\kappa_2(V)$, distribution of evals, etc..

pre-conditioning: transform system for better properties

$$Ax = b \Rightarrow M^{-1}Ax = M^{-1}b$$

$M = I$ $\qquad$ $M = A \quad A^{-1}Ax = A^{-1}b \quad Ix = A^{-1}b$

some pre-computation
much better system

$C_L, C_R \qquad C_L C_R \approx A \qquad C_L^{-1} A C_R^{-1} y = C_L^{-1} b \qquad C_L^{-1} A z = C_L^{-1} b$

$C_R z = y$

$$\left.\begin{aligned} D_r &= \text{row sums} = A\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} \\ D_c &= \text{col sums} = A^T\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\end{aligned}\right\} \text{two mat-vecs}$$

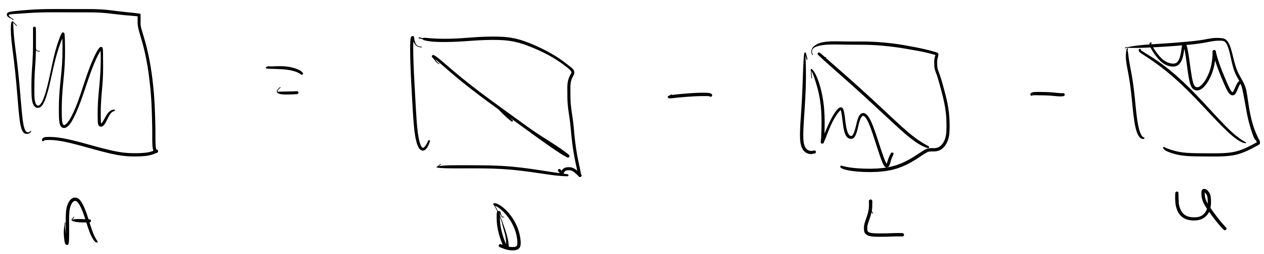
$$D_r^{-1} A D_c^{-1} y = b$$

$$A\begin{pmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{pmatrix} = Ae_i \rightarrow v \rightarrow e_i^T v = e_i^T A e_i = A_{ii}$$

$$Ax = b \Rightarrow D^{-1}Ax = D^{-1}b$$

$$A = M - N \qquad M^{-1}(M-N)x = M^{-1}b \qquad Mx_{k+1} = Nx_k + b$$

$$\underline{(I - M^{-1}N)x = M^{-1}b} \qquad x_{k+1} = M^{-1}Nx_k + M^{-1}b$$

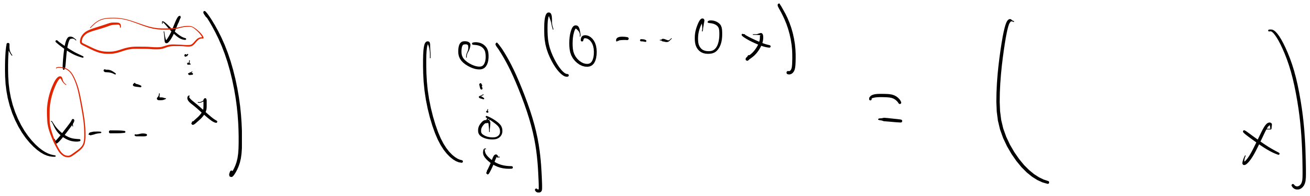


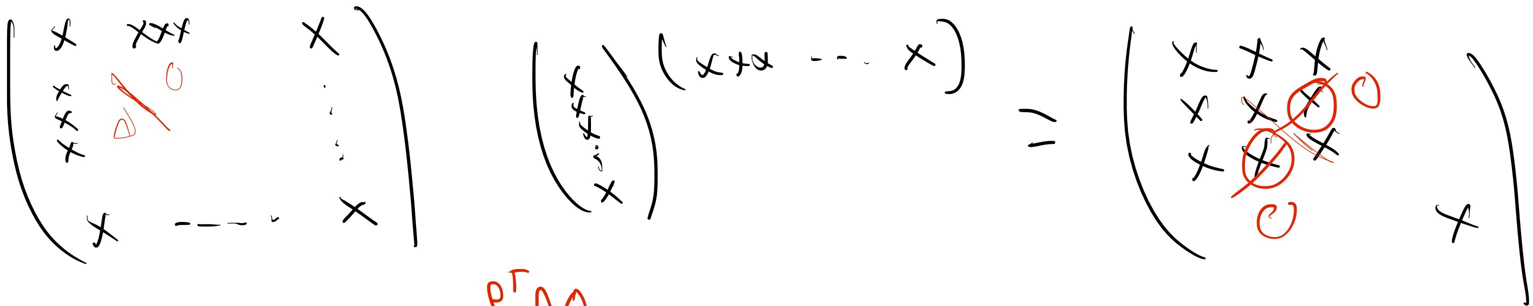
Jacobi: $M = D$, $N = L + U$ $\Rightarrow$ diagonal scaling

Gauss-Seidel: $M = D - L$, $N = U$

sparse direct $\quad Ax = b$

$$P^T A Q = LU$$



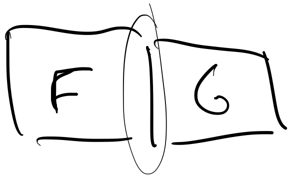


$$M = LU \qquad M^{-1} \overset{P^TAQ}{\cancel{A}}\, x = M^{-1} b$$

Incomplete LU: only fill in where P^TAQ has non-zeros

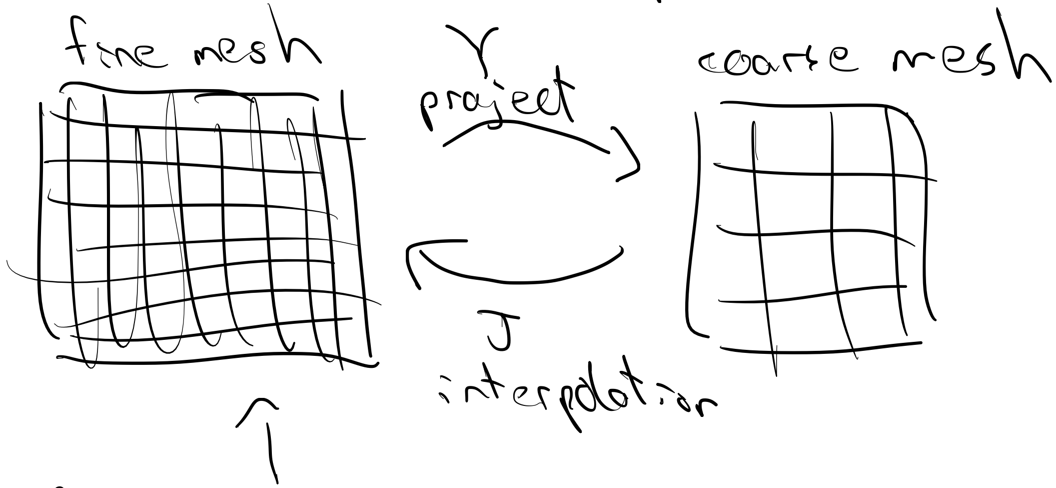
~ same storage cost (2x), can work extremely well

Physics-based problems



$$A = \begin{pmatrix} A_{FF} & A_{FG} \\ A_{GF} & A_{GG} \end{pmatrix} \qquad D = \begin{pmatrix} A_{FF} & 0 \\ 0 & A_{GG} \end{pmatrix}$$ block Jacobi

"Domain decomposition"



$M \approx A_f$

$M \approx J A_c Y$

1. projects
2. solve on coarse mesh
3. interpolates to fine mesh

"multigrid"