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$Ax = b \qquad K_k(A, b) = \{b, Ab, \ldots, A^{k-1} b\}$

$\beta_0 = \|b\|_2$

$q_1 = b, \quad q_0 = 0$

$$A Q_k = Q_{k+1} \bar{T}_k \qquad \bar{T}_k = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{k-1} \\ & & \beta_{k-1} & \alpha_k \\ & & & \beta_k \end{pmatrix}$$

(1) $\beta_k q_{k+1} = A q_k - \alpha_k q_k - \beta_{k-1} q_{k-1}$

(2) $Q_k^T A Q_k = T_k = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{k-1} \\ & & \beta_{k-1} & \alpha_k \end{pmatrix}$

A SPD

	opt. problem	subproblem	factorization	recurrences
CG	$\min \|Ax - b\|^2_{A^{-1}}$ s.t. $x \in K_k$	$T_k y_k = \beta_0 e_1$ $x_k = Q_k y_k$	$T_k = L_k D_k L_k^T$	$W_k L_k^T = Q_k$ $L_k D_k z_k = \beta_0 e_1$ $x_k = x_{k-1} + \zeta_k w_k$
MINRES	$\min \|Ax - b\|_2^2$ s.t. $x \in K_k$	$\min \|\bar{T}_k y_k - \beta_0 e_1\|_2$ $x_k = Q_k y_k$	$\bar{T}_k = G_k \begin{pmatrix} R_k \\ 0 \end{pmatrix}$ $G_k^T G_k = I$ $R_k = (\ )$	$R_k y_k = z_k$ $W_k R_k = Q_k$ $x_k = x_{k-1} + \zeta_k w_k$

$\min \|Ax - b\|_2$ s.t. $x \in K_k$

$x = Q_k y_k$

$\|A Q_k y_k - b\|_2^2$

$= \|Q_{k+1} \bar{T}_k y_k - b\|_2^2$ (Q_{k+1}^T)

$= \left\| \begin{pmatrix} I \\ 0 \end{pmatrix} \bar{T}_k y_k - \begin{pmatrix} \beta_0 \\ 0 \end{pmatrix} \right\|_2^2$

$= \|\bar{T}_k y_k - \beta_0 e_1\|_2^2$

LLS: $\underset{k+1}{\bar{T}_k}^{\,k} = \underset{k+1}{G_k}^{\,k+1} \begin{pmatrix} R_k \\ 0 \end{pmatrix}^{k}_{1} \quad G_k^T G_k = I$

$= \|G_k^T(\bar{T}_k y_k - \beta_0 e_1)\|_2^2$

$= \left\| \begin{pmatrix} R_k \\ 0 \end{pmatrix} y_k - \begin{pmatrix} z_k \\ \bar{\eta}_{k+1} \end{pmatrix} \right\|_2^2$

$R_k y_k = z_k$

$= \left\| \begin{pmatrix} 0 \\ \bar{\eta}_{k+1} \end{pmatrix} \right\|_2^2$

$\bar{T}_4$

$$\tilde{G}_4^T \tilde{G}_3^T \tilde{G}_2^T \tilde{G}_1^T \begin{pmatrix} \times & \times & \cancel{\times} & & \\ \cancel{0} & \times & \times & \cancel{\times} & \\ & \cancel{0} & \times & \times & \\ & & \cancel{0} & \times & \times \\ & & & \cancel{0} & \times \end{pmatrix} \longrightarrow \begin{pmatrix} R_4 \\ 0 \end{pmatrix}$$

$$\tilde{G}_4^T \tilde{G}_3^T \tilde{G}_2^T \tilde{G}_1^T \begin{pmatrix} \beta_0 e_1 \\ \cancel{0} \\ \cancel{0} \\ \cancel{0} \\ \cancel{0} \end{pmatrix} \longrightarrow \begin{pmatrix} z_4 \\ \bar{\eta}_5 \end{pmatrix}$$

$$\tilde{G}_{k+1}^T \tilde{G}_k^T \tilde{G}_{k-1}^T \cdots \tilde{G}_1^T \bar{T}_{k+1} = \begin{pmatrix} \bar{T}_k & \begin{matrix} \times \\ \times \\ \times \end{matrix} \end{pmatrix} \longrightarrow \begin{pmatrix} R_k & \begin{matrix} \cancel{\times} \\ \cancel{\times} \end{matrix} \\ 0 & 0 \end{pmatrix}$$

$$\tilde{G}_{k+1}^T \begin{pmatrix} z_k \\ \bar{\eta}_{k+1} \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} z_k \\ \eta_{k+1} \\ \bar{\eta}_{k+2} \end{pmatrix} = \begin{pmatrix} z_{k+1} \\ \bar{\eta}_{k+2} \end{pmatrix}$$

$W_k R_k = Q_k$

$W_k \begin{pmatrix} \diagdown & \begin{matrix} c \\ b \\ a \end{matrix} \end{pmatrix} e_k = Q_k e_k$

$a w_k + b w_{k-1} + c w_{k-2} = q_k$

① $x_k = Q_k y_k$ ② $R_k y_k = z_k$

③ $W_k R_k = Q_k$

$$
\begin{aligned}
x_k &= W_k R_k y_k \\
&= W_k z_k \\
&= \begin{bmatrix} W_{k-1} & w_k \end{bmatrix} \begin{pmatrix} z_{k-1} \\ \eta_k \end{pmatrix} \\
&= W_{k-1} z_{k-1} + \eta_k w_k \\
&= x_{k-1} + \eta_k w_k
\end{aligned}
$$

$$w_k = \frac{1}{a}\left(q_k - b\, w_{k-1} - c\, w_{k-2}\right)$$

MINRES

for $k = 2, 3, \ldots$

$$
\text{Lanczos}\left\{
\begin{aligned}
v &= A q_k \\
\alpha_k &= q_k^T v \\
\beta_k q_{k+1} &= v - \alpha_k q_k - \beta_{k-1} q_{k-1}
\end{aligned}
\right.
$$

$$\tilde{G}_k^T \tilde{G}_{k-1}^T \tilde{G}_{k-2}^T \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta_{k-1} \\ \alpha_k \\ \beta_k \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ \vdots \\ 0 \\ c \\ b \\ a \\ 0 \end{pmatrix}$$

$$\tilde{G}_k \begin{pmatrix} \bar{\eta}_k \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} \eta_k \\ \bar{\eta}_{k+1} \end{pmatrix}$$

$$x_k = x_{k-1} + \eta_k w_k$$

if $|\bar{\eta}_{k+1}| < tol$

return x_k

Residuals

$$r_k = b - Ax_k$$

$$= \beta_0 q_1 - AQ_k y_k$$

$$= \beta_0 Q_{k+1} e_1 \quad Q_{k+1} \bar{T}_k y_k$$

$$= Q_{k+1} \underbrace{\left(\beta_0 e_1 - \bar{T}_k y_k\right)}_{t_{k+1}}$$

$$r_k = Q_{k+1} t_{k+1}$$

$$\|r_k\|_2 = \|t_{k+1}\|_2$$

MINRES: making $\|t_{k+1}\|_2$ small as possible

CG: $r_k \propto q_{k+1}$

$$t_{k+1} \propto e_{k+1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x \end{pmatrix}$$

Error analysis

$$r_k = \min_x \|b - Ax\| \text{ s.t. } x \in K_k$$

$$x = p(A)b \quad p \in P_{k-1}$$

$$\Rightarrow \|b - Ap(A)b\|_2$$

$$= \|(I - p(A))b\|_2$$

$$\Rightarrow \min_q \|q(A)b\|_2 \le \|q(A)\|_2 \overbrace{\|b\|_2}^{\|r_0\|_2}$$

$$\text{s.t. } q \in P_k \quad q(0) = 1$$

$$\frac{\|r_k\|}{\|r_0\|} \le \|q(A)\|_2$$

$$A = V\Lambda V^T \quad \|q(A)\|_2 = \|q(\Lambda)\|_2 = \max_{\lambda_i} |q(\lambda_i)|$$

$$\min_q \max_{\lambda_i} |q(\lambda_i)|$$

$$\text{s.t. } q \in P_k$$

$$q(0) = 1$$