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Problem: $Ax = b$ $\left(\min_x \|Ax - b\|_2^2 \text{ and } Ax = \lambda x \text{ later} \right)$

$$\begin{array}{ccccc} & n & & 1 & \\ n & \boxed{A} & & \boxed{x} & = \boxed{b} \end{array}$$

Iterative solvers: only have access to Ax, $A^T x$ (not A_{ij})

Idea: search for solution in a subspace ($\subseteq \mathbb{R}^n$)

Krylov subspace: $K_k(A, b) = \operatorname{span}(\{b, Ab, A^2 b, A^3 b, \dots, A^{k-1} b\})$

Method: $\min_x \|Ax - b\|$ $\quad \begin{array}{cc} & n \\ n & \boxed{A} \end{array} \quad \begin{array}{cc} & 1 \\ n & \boxed{b} \end{array}$

s.t. $x \in K_k(A, b)$

Overall plan:

$\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$

$\tilde{x}_j \in K_j(A, b)$

$\tilde{x}_{j+1}$ from $\tilde{x}_j$

Basis: $M = [b, Ab, \dots, A^{k-1} b]$

$$\min_z \|\underbrace{AM}z - b\|$$

write $x = Mz$ $\quad \|Fz - b\|_2 \Rightarrow$ least squares (MINRES)

$$\begin{array}{ccccc} & k & & 1 & \\ n & \boxed{} & k & \boxed{} & - \boxed{} \end{array}$$

Approx. theory

$$K_k = \text{span}\{A^0 b, A^1 b, \dots, A^{k-1} b\}$$

$$x \in K_k \Rightarrow x = c_0 A^0 b + c_1 A^1 b + \dots + c_{k-1} A^{k-1} b$$
$$= p(A) b$$

$$\min_x \|Ax - b\| \quad \text{s.t. } x \in K_k(A,b) \iff \min_p \|p(A)b - b\| \quad \text{s.t. } p \in P_{k-1}$$

$$A = V \Lambda V^{-1} \qquad p(A) = V p(\Lambda) V^{-1}$$

$$\Rightarrow \min_{p \in P_{k-1}} \|V p(\Lambda) V^{-1} b - b\|$$
$$= \|V(p(\Lambda) V^{-1} b - V^{-1} b)\|$$
$$= \|V(p(\Lambda) - I) V^{-1} b\|$$
$$\le \underbrace{\|V\| \|V^{-1}\|}_{\kappa(V)} \underbrace{\|b\|}_{RHS} \underbrace{\|p(\Lambda) - I\|}$$

$$\min \|q(\Lambda)\| \quad \text{s.t. } q \in P_{k-1},\ q(0) = -1$$

$[b, Ab, \dots, A^{k-1} b]$

"power basis"

orthog basis

Idea: Orthonormal basis

Gram–Schmidt

$h_{10} = \|b\|$

$Q_k = [q_1 \cdots q_k]$ $\quad h_{10} q_1 = b$

$\mathrm{span}(A q_1) = \mathrm{span}(A b)$

$q_1^T A q_1 = q_1^T (h_{11} q_1 + h_{21} q_2)$

$h_{12} = q_1^T A q_1$

$h_{2,2} q_2 = A q_1 - h_{2,1} q_1$

$\vdots$

$q_s^T A q_k = q_s^T \left(\sum_{j=1}^{k+1} h_{j,k} q_j\right)$ $\quad h_{s,k}$ $\quad 1 \le s \le k$

$\Longleftrightarrow$

$h_{k+1,k} q_k = A q_k - \sum_{j=1}^{k} h_{j,k} q_j$

Arnoldi decomposition

$A Q_k = Q_{k+1} \bar{H}_k$

$$\bar{H}_k = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1k} \\ h_{21} & h_{22} & & \vdots \\ & h_{32} & & \vdots \\ & & & h_{k,k} \\ & 0 & & h_{k+1,k} \end{pmatrix} \in \mathbb{R}^{k+1 \times k}$$

$A Q_k e_1 = Q_{k+1} \bar{H} e_1$

$A q_1 = h_{11} q_1 + h_{21} q_2$

$A Q_k e_k = Q_{k+1} \bar{H}_k e_k$

$A q_k = \sum_{j=1}^{k+1} h_{j,k} q_j$

could use MGS

$[b, Ab, \ldots, A^{k-1} b]$

Arnoldi process

get q_{k+1} from A, Q_k

What if A is symmetric?

$$AQ_k = Q_{k+1}\bar{H}_k$$

$$Q_k^T A Q_k = Q_k^T Q_{k+1}\begin{pmatrix} H_{k-1} & h_k \\ 0 & h_{k+1,k}\end{pmatrix}$$

$$= [I \quad 0]\begin{pmatrix} & \\ & \end{pmatrix}$$

$$= \underbrace{[H_{k-1} \quad h_k] = \begin{pmatrix} \ddots & \\ 0 & \end{pmatrix}}_{= H_k \text{ with last row omitted}}$$

$Q_k^T A Q_k$ symm

$$\Rightarrow Q_k^T A Q_k = \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$$

$$AQ_k = Q_{k+1}\bar{T}_k$$

$$\mathbb{R}^{k+1\times k} \ni \bar{T}_k = \begin{pmatrix} \alpha_1 & \beta_1 \cdots & \beta_{k+1} \\ \beta_1 \cdots & \beta_{k-1} & \alpha_k \\ & & \beta_k \end{pmatrix} = \begin{pmatrix} \ddots & \\ & \cdot \end{pmatrix}$$

Lanczos decomposition

Lanczos process

$$AQ_k = Q_{k+1}\bar{T}_k$$

$$AQ_k e_k = Q_{k+1}\bar{T}_k e_k$$

$$A \cdot q_k = \left.\begin{aligned} &\beta_{k-1} q_{k-1} \\ &+ \alpha_k q_k \\ &+ \underline{\beta_k}\ \underline{q_{k+1}} \end{aligned}\right\} \text{3-term recurrence}$$

Input: A, b

$q_0 = 0$, $\beta_0 q_1 = b$, $\beta_0 = \|b\|_2$

for $k = 1, 2, \ldots$

- $v = Aq_k$ n storage
- $\alpha_k = q_k^T v$ $\Rightarrow \alpha_k$
- $\beta_k q_{k+1} = [v - \alpha_k q_k - \beta_{k-1} q_{k-1}]$

$[b, Ab, \dots, A^{k-1}b] \to Q_k R_k$ (not explicitly computing)

$$AQ_k = Q_{k+1}\bar{H}_k \qquad Q_k^T A Q_k = H_k \qquad \bar{H}_k = \begin{array}{c} \\ k \\ 1 \end{array} \begin{array}{c} \overbrace{\qquad\qquad}^{k} \\ \left(\begin{array}{cc} \multicolumn{2}{c}{H_k} \\ 0 & h_{k+1,k} \end{array}\right) \end{array}$$

$$AQ_k = Q_{k+1}\bar{T}_k \qquad Q_k^T A Q_k = T_k \qquad \bar{T}_k = \begin{array}{c} \\ k \\ 1 \end{array} \begin{array}{c} \overbrace{\qquad\qquad}^{k} \\ \left(\begin{array}{cc} \multicolumn{2}{c}{T_k} \\ 0 & \beta_k \end{array}\right) \end{array}$$

Problems: MGS not stable

Arnoldi – re-orthogonalize all cols every so often

Lanczos – fancier techniques

"partial re-orthog"