

Nov 11, 2020

Problem: $Ax = b$ ($\min_x \|Ax - b\|_2^2$ and $Ax = b$ later)

$$\boxed{A} \cdot \boxed{x} = \boxed{b}$$

Iterative solvers: only have access to Ax , $A^T x$ (not A_{ij})

Idea: search for solution in a subspace ($\subseteq \mathbb{R}^n$)

Krylov subspace: $K_k(A, b) = \text{span}(\{b, Ab, A^2b, A^3b, \dots, A^{k-1}b\})$

Method: $\min_x \|Ax - b\|$ $\cdot \boxed{A} \cdot \boxed{b}$
s.t. $x \in K_k(A, b)$

Basis: $M = [b, Ab, \dots, A^{k-1}b]$

$$\min_z \|AMz - b\|$$

write $x = Mz$ $\| \begin{bmatrix} A \\ I \end{bmatrix} z - b \|_2 \Rightarrow$ least squares (MINRES)

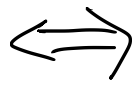
Overall plan:
 $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$
 $\tilde{x}_j \in K_j(A, b)$
 \tilde{x}_{j+1} from \tilde{x}_j

Approx. theory

$$K_k = \text{span} \{ A^0 b, A^1 b, \dots, A^{k-1} b \}$$

$$x \in K_k \Rightarrow x = c_0 A^0 b + c_1 A^1 b + \dots + c_{k-1} A^{k-1} b \\ = p(A) b$$

$$\min_x \|Ax - b\| \\ \text{s.t. } x \in K_k(A, b)$$



$$\min_p \|p(A) b - b\| \\ \text{s.t. } p \in \mathcal{P}_{k-1}$$

$$A = V \Lambda V^{-1} \quad p(A) = V p(\Lambda) V^{-1}$$

$$\Rightarrow \min_{p \in \mathcal{P}_{k-1}} \|V p(\Lambda) V^{-1} b - b\|$$

$$= \|V(p(\Lambda) V^{-1} b - V^{-1} b)\|$$

$$= \|V(p(\Lambda) - I) V^{-1} b\|$$

$$\leq \underbrace{\|V\|}_{\mathcal{K}(V)} \underbrace{\|V^{-1}\|}_{\text{RHS}} \|b\| \|p(\Lambda) - I\|$$

$$\min \|q(\Lambda)\| \\ \text{s.t. } q \in \mathcal{P}_{k-1}, q(0) = -1$$

$$[b, Ab, \dots, A^{k-1} b]$$

"power basis"

orthog basis

Idea: Orthogonal basis

Gram-Schmidt

$$h_{10} = \|b\|$$

$$Q_k = [q_1 \dots q_k] \quad h_{10} q_1 = b$$

$$\text{span}(A q_1) = \text{span}(A b)$$

$$q_1^T A q_1 = q_1^T (h_{11} q_1 + h_{21} q_2)$$

$$h_{12} = q_1^T A q_1$$

$$h_{2,2} q_2 = A q_1 - h_{2,1} q_1$$

...

$$q_s^T (A q_k) = \sum_{j=1}^{k+1} h_{j,k} q_j^T q_s \quad h_{s,k} \quad 1 \leq s \leq k$$

$$h_{k+1,k} q_k = A q_k - \sum_{j=1}^k h_{j,k} q_j$$

Arnold: decomposition

$$A Q_k = Q_{k+1} \bar{H}_k$$

$$\bar{H}_k = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1k} \\ h_{21} & h_{22} & & \vdots \\ & h_{32} & & h_{k,k} \\ & & & \ominus h_{k+1,k} \end{pmatrix} \in \mathbb{R}^{(k+1) \times k}$$

$$A Q_k e_1 = Q_{k+1} \bar{H}_k e_1$$

$$A q_1 = h_{11} q_1 + h_{21} q_2$$

$$A Q_k e_k = Q_{k+1} \bar{H}_k e_k$$

$$A q_k = \sum_{j=1}^{k+1} h_{j,k} q_j$$

could use MGS

$$[b, A b, \dots, A^{k+1} b]$$

Arnoldi process

get q_{k+1} from A, Q_k

$$[b, Ab, \dots, A^{k-1}b] \rightarrow Q_k R_k \quad (\text{not explicitly computing})$$

$$AQ_k = Q_{k+1} \bar{A}_k \quad Q_k^T A Q_k = H_k$$

$$\bar{A}_k = \begin{pmatrix} H_k & \\ & \underbrace{0 \quad h_{k+1,k}}_{\substack{\text{---} \\ \uparrow \\ \text{---}}} \end{pmatrix}$$

$$AQ_k = Q_{k+1} \bar{T}_k \quad Q_k^T A Q_k = \bar{T}_k$$

$$\bar{T}_k = \begin{pmatrix} \underbrace{\bar{T}_k} & \\ & \underbrace{0 \quad \beta_k}_{\substack{\text{---} \\ \uparrow \\ \text{---}}} \end{pmatrix}$$

Problems: MGS not stable

Arnoldi - re-orthogonalize all cols every so often

Lanczos - fancier techniques
"partial re-orthog"