Nov 4, 2020

$$\underset{n\times n}{A}\,x = b \qquad \min_x \| \underset{m\times n}{A}\,\underset{n\times 1}{x} - \underset{m\times 1}{b} \|_2 \qquad \underset{n\times n}{A}\,\underset{n\times 1}{x} = \lambda \underset{n\times 1}{x}$$

$PA = LU$ $\qquad A = QR$ $\qquad A = QTQ^H$

$L(Ux) = P^T b$ $\qquad Rx = Q^T b$ $\qquad \lambda = T_{ii}$, $Ty = \lambda y$

$x = Qy$

$\boxed{\text{direct methods}}$

$A \to$ factorization $\to$ easy $\quad O(n^3) \quad O(mn^2)$

benefits: ① can be fast (n small)

② reliable: stability

③ reuse! $Ax = c \Rightarrow LUx = P^T c \quad O(n^2)$

Mostly "black box" performance

$A, b \to \boxed{Ax=b} \to x$ $\quad O(n^3)$

$A, b \to \boxed{LS} \to \hat{x}$ $\quad O(mn^2)$

$A \to \boxed{\text{eval evec}} \to \lambda_1, \ldots, \lambda_n$; $v_1, \ldots, v_n$ $\quad O(n^3)$

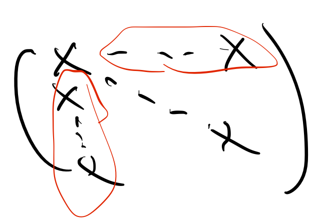
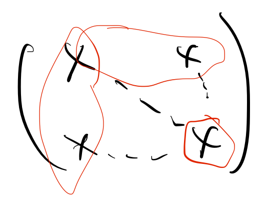
Not totally black boxes

1. reuse
2. $\kappa_2(A) = \|A\|_2 \|A^{-1}\|$ on sensitivity
   $Ax = b$, LLS
3. whether or not LLS is "good"
   $A^T b \approx 0 \iff b \perp \text{Range}(A) \Rightarrow$ no "good" solution
4. $$\min_x \left\| \underset{m \times n}{A}\, \underset{n \times 1}{x} - \underset{m \times 1}{b} \right\|_2^2 + \lambda \|x\|_2$$

$$\iff \min_x \left\| \underset{m+n}{\begin{pmatrix} A \\ \sqrt{\lambda} I \end{pmatrix}} x - \begin{pmatrix} b \\ 0 \end{pmatrix} \right\|_2^2 \quad O((m+n)n^2) = O(n^3)$$

$$\lambda \to 0 \quad ((A^T)^+)^T b \quad O(mn^2)$$

5. sparse A, $Ax = b$
   $PAQ = LU$

 vs. 

Sparse matrices cause problems for direct methods

Example: English Wikipedia (2013) 4.2M pages 101M links

Compressed sparse column (CSC)

col 2
in indices
4 to 6

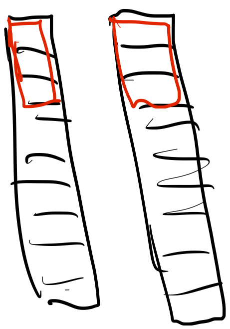
$2 \cdot 101M \cdot 8$ bytes per value

≈ 1.5 GB

LU factors $(4.2M)^2 \cdot 8$ bytes per entry

≈ 128 TB

One fast thing: $y = Ax = \sum_j A(:,j)\, x_j$



$A_{10,1} = 1$

$A_{1358,1} = 1$

$A_{400221,1} = 1$

$y = 0$

$y_{10} \mathrel{+}= x_1$

$y_{1358} \mathrel{+}= x_1$

$y_{400221} \mathrel{+}= x_1$

$O(\mathrm{nnz}(A))$

Black box for iterative solvers

$x \to \boxed{\text{matvec}} \to Ax \qquad x \to \boxed{} \to A^T x \quad (x^T A)$

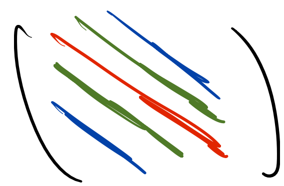
1. A sparse $\quad Ax$ in $O(\mathrm{nnz}(A))$ time

2. sparse + low rank

$$\underset{n}{\overset{n}{\boxed{A}}} = \underset{n}{\overset{n}{\boxed{S}}} + \underset{n}{\overset{k}{\boxed{L}}}\;{}_{k}\overset{n}{\boxed{R}} \qquad Ax = \underbrace{Sx}_{\mathrm{nnz}(S)} + \underbrace{L(Rx)}_{O(nk)}$$

3. Low displacement rank

Toeplitz: $T_{ij} = t_{i-j}$



$O(n \log n)$

4. Kernel $\quad K_{ij} = \dfrac{1}{\|x_i - x_j\|_2^2} \quad x_1, \dots, x_n \in \mathbb{R}^2$ quasi-uniformly $\quad O(n)$ (approx)

5. Product structure

$$A \otimes B = \underbrace{\begin{pmatrix} a_{11}B & \cdots & a_{n1}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nn}B \end{pmatrix}}_{n^2 \times n^2} \quad O(n^4)$$

$$\mathrm{vec}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}$$

$$(A \otimes B)\,\mathrm{vec}(C) = \mathrm{vec}(\underbrace{BCA^T}_{O(n^3)})$$

Iterative solvers

1. Splitting / stationary

2. Krylov subspace methods

$$\underbrace{b, \; Ab, \; A^2 b, \; A^3 b}, \; \ldots$$

3. Geometric methods (multigrid)