Nov 2, 2020

Problem with repeated evals

$$A(\varepsilon) = \begin{pmatrix} \lambda & 1 \\ \varepsilon & \lambda \end{pmatrix}$$

$$p_\varepsilon(z) = \det(A(\varepsilon) - zI) = (\lambda - z)^2 - \varepsilon$$
$$= z^2 - 2\lambda z + \lambda^2 - \varepsilon$$
$$= (z - (\lambda + \sqrt{\varepsilon}))(z - (\lambda - \sqrt{\varepsilon}))$$

evals $= \lambda \pm \sqrt{\varepsilon}$ continuous func of ε
but not diff

$\varepsilon \to 0$ $A(0) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ Jordan block

$O(\varepsilon)$ perturbation $\Rightarrow$ $\sqrt{\varepsilon}$ change cond number $= \infty$

$$s\left\{\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ \varepsilon & & & \lambda \end{pmatrix}\right. \quad \text{evals} = z + O(\varepsilon^{1/s})$$

Better perturbation theory for unsymm. matrices

$Ax = \lambda x$, λ simple, $\|x\|_2 = 1$ (w.r.t. $\|\cdot\|$ over $\mathbb{C}$)

$S^{-1}AS = \mathrm{diag}(J_1, \dots, J_k)$ $\quad J_i = s\left\{\begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}\right.$ $\quad J_i e_1 = \lambda_i e_1$

$AS = S\,\mathrm{diag}(J_1, \dots, J_k)$ $\quad e_s^T J_i = \lambda_i e_s^T \quad e_s^T e_1 = 0$

$S^{-1}A = \mathrm{diag}(J_1, \dots, J_k)\, S^{-1}$

Left evecs: $y^H A = \lambda y^H$, $\|y\|_2 = 1$ (w.r.t. $\|\cdot\|$ over $\mathbb{C}$)

$S^{-1}AS = \Lambda \implies$ right evecs given by cols of S
left evecs given by rows of S^{-1}

$$(A + \delta A)(x + \delta x) = (\lambda + \delta\lambda)(x + \delta x)$$

$$Ax = \lambda x$$

$$y^H(A\delta x + \delta A x) = y(\lambda \delta x + \delta\lambda x) + \text{higher-order terms}$$

$$y\, A\delta x + y \cdot \delta A x = \lambda y^H \delta x + \delta\lambda \cdot x$$

$$y^H \delta A x = \delta\lambda\, y^H x$$

$$\delta\lambda = \frac{y^H \delta A x}{y^H x} + \text{higher-order}$$

$$|\delta\lambda| \le \frac{\|\delta A\|_2}{|y^H x|} + O(\|\delta A\|^2)$$

① Symm $y^H = x^T$

② $\to$ Jordan block (nontrivial) $\to \infty$

Last time: Bauer-Fike $\quad X^{-1}AX = \Lambda \quad \mu \in \lambda(A+E)$

$$\min_j |\mu - \lambda_j| \le \|X\|_p \|X^{-1}\|_p \|E\|_p$$

Evals of $A+E \in \bigcup_j D_{\lambda_j}\left(\|X\|\,\|X^{-1}\|\,\|E\|\right) \qquad \|X^{-1}EX\|_\infty$

Better: $\qquad \in \bigcup_j D_{\lambda_j}\left(n\|E\|_2 / |y_j^H x_j|\right)$

Gershgorin's: evals of $M \in \bigcup_j D_{M_{jj}}\left(\sum_{i \ne j} |M_{ij}|\right)$

$$X^{-1}(A+E)X = \Lambda + \underbrace{X^{-1}EX}_{F} \qquad |\mu - \lambda_j| \le \sum_{i\ne j} |f_{ji}| \le \overbrace{\sqrt{n}\,\|F(j,:)\|_2}^{C-S}$$

$$\|F(j,:)\|_2 = \|e_j^T X^{-1} E X\|_2 \le \|e_j^T X^{-1}\|_2 \|E\|_2 \overbrace{\|X\|_2}^{\sqrt{n}}$$

choose X so that cols have unit norm $\Rightarrow \|X\|_2 \le \sqrt{n}$

rows of X^{-1} are left evecs $\Rightarrow X^{-1} = \begin{bmatrix} c_1 y_1^H \\ \vdots \\ c_n y_n^H \end{bmatrix} \quad \|y_j\|_2 = 1$

$(X^{-1}X) = I$

$$1 = (X^{-1}X)_{jj} = c_j y_j^H x_j \Rightarrow c_j = \frac{1}{y_j^H x_j}$$

$$\Rightarrow \|e_j^T X^{-1}\|_2 = \left|\frac{1}{y^H x}\right|$$

$$A = [Q_1\ Q_2]\begin{bmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{bmatrix}\begin{bmatrix}Q_1^T\\ Q_2^T\end{bmatrix}$$

E symm

$$A+E = [\tilde{Q}_1\ \tilde{Q}_2]\begin{bmatrix}\tilde{\Lambda}_1 & 0\\ 0 & \tilde{\Lambda}_2\end{bmatrix}\begin{bmatrix}\tilde{Q}_1^T\\ \tilde{Q}_2^T\end{bmatrix}$$

$$\Lambda_1 \subseteq [a, b]$$

$$\delta = \min_k \left\{ |x - (\tilde{\Lambda}_2)_{kk}| : x \in [a,b] \right\}$$

$$\|\tilde{Q}_2^T Q_1\| \le \|\tilde{Q}_2^T E Q_1\|/\delta \le \|E\|/\delta$$

$= d(\text{range}(Q_1), \text{range}(\tilde{Q}_1))$

$= \|\sin\Theta\|$

$\tilde{Q}_2^T \perp \text{range}(Q_1)$

Proof:

$$AQ_1 = Q_1\Lambda_1$$

$$EQ_1 = (A+E-A)Q_1 = (A+E)Q_1 - Q_1\Lambda_1$$

$$\tilde{Q}_2^T(A+E) = \tilde{\Lambda}_2\tilde{Q}_2^T$$

$$\tilde{Q}_2^T E Q_1 = \tilde{Q}_2^T[(A+E)Q_1 - \Lambda_1 Q_1]$$

$$= \tilde{\Lambda}_2 \underbrace{\tilde{Q}_2^T Q_1}_{M} - \underbrace{\tilde{Q}_2^T Q_1}_{M}\Lambda_1$$

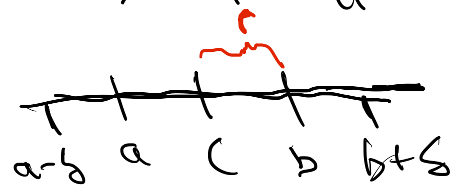
$$= \tilde{\Lambda}_2 M - M\Lambda_1$$

$$\|\tilde{\Lambda}_2 M - M\Lambda_1\|$$

$$= \|(\tilde{\Lambda}_2 - cI)M - M(\Lambda_1 - cI)\|$$

$$\ge \|(\tilde{\Lambda}_2 - cI)M\| - \|M(\Lambda_1 - cI)\|$$

$$c = (a+b)/2$$



$$\|(\tilde{\Lambda}_2 - cI)M\| \ge \left(\min_k |(\tilde{\Lambda}_2)_{kk} - c|\right)\|M\| \ge (r+\delta)\|M\|$$

$$\le \|M\|\,\|\Lambda_1 - cI\| \le \|M\|\, r$$

$$\|\tilde{Q}_2^T E Q_1\| \ge (r+\delta)\|M\| - r\|M\|$$

$$= \delta\|\tilde{Q}_2^T Q_1\|$$