Oct 30, 2020

$A = U \Sigma V^T$

$A^T A = V \Sigma U^T U \Sigma V^T = V \Sigma^2 V^T$

Idea: 1 Unstable!

1. form $A^T A = M$
2. use eigensolver with M
3. $A^T U = V \Sigma$

$$\Big| \underbrace{\lambda_k(M+E)}_{\tilde{\lambda}_k} - \lambda_k(M) \Big| \le \|E\|_2 \implies |\tilde{\lambda}_k - \lambda_k| = O(\varepsilon_{mach} \|M\|_2)$$

$\|M\|_2 = \|A\|_2^2$

$$|\tilde{\sigma}_k - \sigma_k| \approx \frac{|\tilde{\lambda}_k - \lambda_k|}{\sqrt{\lambda_k}} = O(\varepsilon \|M\|_2 / \sigma_k) = O(\varepsilon_{mach} \sigma_1^2 / \sigma_k)$$

Want: $|\tilde{\sigma}_k - \sigma_k| = O(\varepsilon_{mach} \cdot \|A\|_2)$

$$S = \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & V\Sigma U^T \\ U \Sigma V^T & 0 \end{pmatrix} \begin{pmatrix} V \\ U \end{pmatrix}$$

$$= \begin{pmatrix} V\Sigma \\ U\Sigma \end{pmatrix} = \begin{pmatrix} V \\ U \end{pmatrix} \Sigma$$

$\|S\|_2 = 2 \cdot \|A\|_2$

Idea 2:

1. form $S = \begin{pmatrix} 0 & A^T \\ A & 0 \end{pmatrix}$
2. eigensolver on S

Bidiagonalization

$$U^TA^TAU = T = \qquad T = B^TB, \quad B = \qquad \text{bidiagonal}$$

$$U_1^T \begin{pmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{pmatrix} \rightarrow \begin{pmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & V_1 \end{pmatrix} \rightarrow \begin{pmatrix} x & x & 0 & 0 \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{pmatrix}$$

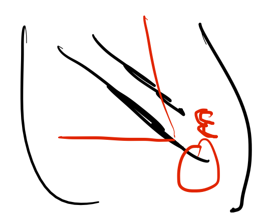
$$\begin{pmatrix} 1 & 0 \\ 0 & U_2^T \end{pmatrix} \rightarrow \begin{pmatrix} x & x & 0 & 0 \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{pmatrix} \quad \cdots\cdots \quad \begin{pmatrix} x & x & 0 & 0 \\ 0 & x & x & 0 \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix}$$

shifted QR $\Rightarrow$ shift $s = T(n,n)$

$$T(n,n) = B^TB(n,n) = e_n^T \, B^T \, B \, e_n$$

$$= B(n-1,n)^2 + B(n,n)^2$$

shift $\sqrt{B(n-1,n)^2 + B(n,n)^2}$



Minimax Theorem

$$A = A^T, \quad \lambda_k = \max_{\dim(S)=k} \min_{0 \neq x \in S} \frac{x^T A x}{x^T x} \quad (*) \qquad \Big| \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$$

Proof: $A = Q \Lambda Q^T, \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

choose $S_k = \operatorname{span}\{q_1, \dots, q_k\}$

$$(*) \geq \min_{0 \neq x \in S_k} \frac{x^T A x}{x^T x} \qquad x = q_1 y_1 + \dots + q_k y_k = [q_1 \dots q_k] y = Q \begin{pmatrix} y \\ 0 \end{pmatrix}$$

y

$$= \frac{y^T Q^T Q \Lambda Q^T Q y}{y^T Q^T Q y} = \frac{\sum_{i=1}^{k} \lambda_i y_i^2}{\sum_{i=1}^{k} y_i^2} \geq \lambda_k$$

Let S_k' be any k-dim subspace $\quad S_k' \cap \underbrace{\operatorname{span}\{q_k, \dots, q_n\}}_{n-k+1} \neq \emptyset$

k

$$\min_{0 \neq x \in S_k'} \frac{x^T A x}{x^T x} \leq \frac{z^T A z}{z^T z} = \frac{\sum_{i=k}^{n} \lambda_i z_i^2}{\sum_{i=k}^{n} z_i^2} \leq \lambda_k$$

$$Qz = Q \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x \\ \vdots \\ x \end{pmatrix}$$

$$(*) \leq \lambda_k \qquad \square$$

Weyl's theorem

A symm with evals $\lambda_1, \dots, \lambda_n$
$A+E$ symm $\tilde\lambda_1, \dots, \tilde\lambda_n$ $\implies |\tilde\lambda_k - \lambda_k| \le \|E\|_2$

Cor 1

QR iteration for evals, $A+E$, $\|E\|_2 = O(\varepsilon_{mach} \|A\|_2)$

$$\implies \underbrace{|\tilde\lambda_k - \lambda_k|}_{|\lambda_k|} \le O(\varepsilon_{mach}\|A\|_2) = \underbrace{O(\varepsilon_{mach} \max(|\lambda_1|, |\lambda_n|))}_{|\lambda_k|}$$

Cor 2

A sing vals $\sigma_1, \dots, \sigma_n$
$A+E$ $\tilde\sigma_1, \dots, \tilde\sigma_n$ $|\tilde\sigma_k - \sigma_k| \le \|E\|_2$

Proof: $\tilde\lambda_k = \max\limits_{\dim(S)=k} \min\limits_{0 \ne x \in S} \dfrac{x^T(A+E)x}{x^Tx}$

$$= \underbrace{\frac{x^TAx}{x^Tx}}_{\lambda_k} + \frac{x^TEx}{x^Tx} \le \|E\|_2 = |\lambda_{max}(E)|$$

$\tilde\lambda_k \le \lambda_k + \|E\|_2$

$A' = A+E$
$E' = -E$
$A'+E' = A$

$\lambda_k \le \tilde\lambda_k + \|E'\|_2$
$= \tilde\lambda_k + \|E\|_2$

Bauer-Fike theorem (unsymm)

$$X^{-1}AX = \Lambda \qquad A+E \quad \mu \text{ eval}$$

$$\min_j |\lambda_j - \mu| \le \|X\|_p \|X^{-1}\|_p \|E\|_p$$

$p = 2$, symm

$\|X\|_2 = \|X^{-1}\|_2 = 1$

$\min_j |\lambda_j - \mu| \le \|E\|_2$

Proof: $\mu = \lambda_j$

$$(\Lambda - \mu I)^{-1} \quad X^{-1}(A+E-\mu I)X = \Lambda - \mu I + X^{-1}EX$$

$$I + \underbrace{(\Lambda - \mu I)^{-1} X^{-1} E X}_{M} \text{ singular}$$

-1 is eval of M

$$|-1| \le \rho(M) \le \|M\|_p \le \underbrace{\|(\Lambda - \mu I)^{-1}\|_p}_{\max_j \frac{1}{|\lambda_j - \mu|} = \frac{1}{\min_j |\lambda_j - \mu|}} \|X\|_p \|X^{-1}\|_p \|E\|_p$$