Oct 28, 2020

$A = A^T$, $A = Q\Lambda Q^T$

Power iteration

$\|x_0\|_2 = 1$ $\qquad |\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$

$y = A x_k$

$x_{k+1} = y / \|y\|_2 \qquad x_k = A^k x_0 / \|A^k x_0\|$

$x_0 = Q z \qquad A^k x_0 = Q \Lambda^k z = \lambda_1^k Q \begin{pmatrix} z_1 \\ z_2 (\lambda_2/\lambda_1)^k \\ \vdots \\ z_n (\lambda_n/\lambda_1)^k \end{pmatrix}$

$A^k x_0 = \lambda_1^k \left[z_1 q_1 + z_2 (\lambda_2/\lambda_1)^k q_2 + \dots + (\lambda_n/\lambda_1)^k q_n \right]$

$\|A^k x_0\| = |\lambda_1|^k \sqrt{z_1^2 + z_2^2 (\lambda_2/\lambda_1)^{2k} + \dots + z_n (\lambda_n/\lambda_1)^{2k}}$

$\mathrm{sign}(\lambda_1) \cdot \mathrm{sign}(z_1) \cdot q_1$

$\|x_k - \pm q_1\|_2 = O\left(\left|\frac{\lambda_2}{\lambda_1}\right|^k \right)$

Rayleigh Quotient Iteration

$\|x_0\|_2 = 1, \quad \lambda_0 = x_0^T A x_0$

$(A - \lambda_k I) y = x_k$

$x_{k+1} = y / \|y\|_2,$

$\lambda_{k+1} = x_{k+1}^T A x_{k+1}$

Suppose $Aq = \lambda q$

$\|x_k - q\| \leq \varepsilon$

$$\lambda_k = \frac{x_k^T A x_k}{x_k^T x_k} = R_A(x_k)$$

$$\nabla R_A(x) = \frac{1}{x^T x}\left(\underbrace{(A+A^T)}_{=2A} x - 2R_A(x)x\right)$$

$\nabla R_A(x^*) = 0 \iff Ax^* = R_A(x^*)\, x^*$

$$\underbrace{R_A(x_k)}_{\lambda_k} = \overbrace{R_A(q)}^{\lambda}$$
$$+ \cancel{\nabla R_A(q)^T (x_k - q)} \quad 0$$
$$+ (x_k - q)^T H (x_k - q) \quad O(\varepsilon^2)$$
$$+ \ldots \quad O(\varepsilon^3)$$

$\Rightarrow |\lambda_k - \lambda| \leq O(\varepsilon^2)$

$$\lambda_2[(A - \lambda_k I)^{-1}] / \lambda_1[(A - \lambda_k I)^{-1}]$$

$$\frac{1}{|\lambda_k - \lambda|}$$

$\|x_{k+1} - q\|_2$

$= O(|\lambda_k - \lambda| \cdot \|x_k - q\|)$

$= O(\varepsilon^3)$

$= O(\|x_k - q\|^3)$

Subspace iteration

$$Z_k = AQ_k$$

$$Q_{k+1}R_{k+1} = Z_k$$

$$|\lambda_p / \lambda_{p+1}|$$

Orthogonal iteration

$$p = n$$

$$Z_k = AQ_k$$

$$Q_{k+1}R_{k+1} = Z_k$$

$$Q_k R_{k+1} = AQ_k$$

$$A = Q_k R_{k+1} Q_k^T$$ Schur form

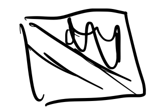
QR iteration

$$T_0 = A$$

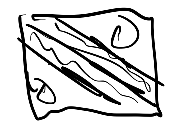
$$Q_k R_k = T_k \qquad R_k = Q_k^T T_k$$

$$T_{k+1} = R_k Q_k = Q_k^T T_k Q_k$$

$$T_0 = U^T A U =$$



$$A^T = A \Rightarrow T_0 = T_0^T =$$



Shifted QRI

$$T_0 = U^T A U$$

$$\sigma_k = T_k(n,n)$$

$$Q_k R_k = T_k - \sigma_k I$$

$$T_{k+1} = Q_k^T T_k Q_k = R_k Q_k + \sigma_k I$$

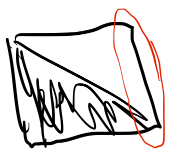
Shifted QR as RQI

$$T - \sigma I = QR$$

$$(T - \sigma I)^{-1} = R^{-1} Q^T$$

$$= Q R^{-T}$$

$$(T - \sigma I)^{-1} R^T e_n = Q e_n$$



$$R^T e_n = r_{nn} e_n$$

$$\sigma = T(n,n) = e_n^T T e_n$$

q_n = one step of RQI with e_n

$$T_{k+1} = Q_k^T T_k Q_k$$

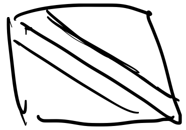
$$T_{k+1}(n,n) = e_n^T Q_k^T T_k Q_k e_n$$

$$= q_n^T T_k q_n$$

$$= \frac{q_n^T T_k q_n}{q_n^T q_n}$$

Get cubic instead of quadratic convergence!

Divide & Conquer

$T = U^T A U =$ 

$$T = \left(\begin{array}{c|c} T_{11} & \cdot \\ \hline \cdot & T_{22} \end{array} \right) \quad \leftarrow \beta$$

$$= \begin{pmatrix} \bar T_{11} & 0 \\ 0 & \bar T_{22} \end{pmatrix} + \underbrace{\begin{pmatrix} \beta & \beta \\ \beta & \beta \end{pmatrix}}_{\beta f f^T}$$

$f^T = (0 \cdots 0\,1\,1\,0 \cdots 0)$

$\bar T_{11} = T_{11} - \beta e_n e_n^T$

$\bar T_{22} = T_{22} + \beta e_1 e_1^T$

$Q_1^T \bar T_{11} Q_1 = D_1$

$Q_2^T \bar T_{22} Q_2 = D_2$

$$\begin{pmatrix} Q_1^T & 0 \\ 0 & Q_2^T \end{pmatrix} T \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}}_{D} + \beta g g^T \qquad g_i \neq 0$$

$g^T = (e_{n/2}^T Q_1,\ e_1^T Q_2)$

$(D + \beta g g^T) q = \lambda q$

$(D - \lambda I) q + \beta g g^T q = 0$

scalars

$g^T q + g^T (D - \lambda I)^{-1} \beta g g^T q = \cancel{0}\ 0$

$g^T q \left(1 + \beta g^T (D - \lambda I)^{-1} g\right) = 0$

$g^T q = 0 \Rightarrow D q = \lambda q \quad q = e_i$

$\Rightarrow g_i \neq 0$

$f(\lambda) = \left(1 + \beta g^T (D - \lambda I)^{-1} g\right)$

roots are the evals!

"secular equation"

$$f(\lambda) = 1 + \beta \sum \frac{g_i^2}{d_i - \lambda}$$