Oct 26, 2020

Double shifted QR

$T_0 = U^T A U = H$

λ_k, σ_k = evals $T_k(n-1:n, n-1:n)$

$Q_k R_k = (T_k - \lambda_k I)(T_k - \sigma_k I)$

$T_{k+1} = Q_k^T T_k Q_k$

check for convergence

$$\left(\begin{array}{c|c} T_{11} & t_{12} \\ \hline 0 & \lambda \end{array}\right) \qquad \left(\begin{array}{c|c} T_{11} & T_{12} \\ \hline 0 & a\ b \\ 0 & c\ d \end{array}\right)$$

"deflation"

$|h_{s+1,s}| = O(\varepsilon_{mach}(|h_{s,s}| + |h_{s+1,s+1}|))$

set $h_{s+1,s} = 0$

$Q_1 R_1 = T - \lambda I$

$T_1 = R_1 Q_1 + \lambda I \quad (T_1 = Q_1^H T Q_1)$

$Q_2 R_2 = T_1 - \sigma I$

$T_2 = R_2 Q_2 + \sigma I \quad (T_2 = Q_2^H T_1 Q_2)$

$(Q_1 Q_2)(R_2 R_1) = \underbrace{(T - \lambda I)(T - \sigma I)}_{M}$

if $\sigma = \bar{\lambda}$,

$M = T^2 - 2\,\mathrm{Real}(\lambda)\, T + |\lambda|^2 I$

- one at a time, complex arith.
- form M explicitly $O(n^3)$ for T^2
- best of both?

Implicit Q theorem

$Q^T A Q = H$ unreduced $h_{k+1,k} \neq 0$

Given q_1,

$q_2, \dots, q_n$ unique up to sign

Proof: $AQ = QH$ $\quad Aq_1 = h_{11} q_1 + h_{21} q_2$

$q_1^T A q_1 = h_{11} \Rightarrow h_{21} q_2 = A q_1 - h_{11} q_1$

$q_s^T A q_i = q_s^T \sum_{j=1}^{i+1} h_{ji} q_j = h_{si} \quad s = 1, \dots, i$

$A q_i - \sum_{j=1}^{i} h_{ji} q_j = h_{i+1,i}\, q_{i+1}$

$M = T_k^2 - 2\,\mathrm{Real}(\lambda)\, T_k + |\lambda|^2 I$

$\underbrace{(Q_1 Q_2)}_{Z}(R_2 R_1) = M$

① $m_1 = M e_1 \quad O(1)$

$m_1^T = (a \ b \ c \ 0 \cdots 0) \quad z_1^T = m_1^T / \|m_1\|$

$\begin{pmatrix} V_1^T & 0 \\ 0 & I \end{pmatrix} m_1 = s \cdot e_1$ (Householder)

$$T_k = \begin{pmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \end{pmatrix}$$

$$\begin{pmatrix} V_1^T & 0 \\ 0 & I \end{pmatrix} T_k \begin{pmatrix} V_1 & 0 \\ 0 & I \end{pmatrix}$$

$$= \quad V_2^T \begin{pmatrix} x & x & x & x & x \\ x & x & x & x & x \\ + & x & x & x & x \\ + & + & x & x & x \\ 0 & + & + & x & x \end{pmatrix} V_2$$

rows 2, 3, 4

sequence $\Rightarrow$ Upper Hess form "bulge chasing"

$V_2, \dots, V_{n-1}$ do not affect first column

$V^T T_k V$ upper Hess $= \tilde{Z}^T T_k \tilde{Z}$

$V = \begin{pmatrix} V_1 & 0 \\ 0 & I \end{pmatrix} V_2 \cdots V_{n-1} \qquad \tilde{Z} = Z \begin{pmatrix} 1 & & & \\ & \pm 1 & & \\ & & \ddots & \\ & & & \pm 1 \end{pmatrix}$

① $\begin{pmatrix} 1 & & & \\ & \pm 1 & & \\ & & \pm 1 & \\ & & & \ddots \\ & & & & \pm 1 \end{pmatrix}$ orthog similarity transform

② can also do this with single shifts

③ never computing R $Q^T T Q$

④ IQT needed "unreduced"; if not, deflate

⑤ We get $A = QTQ^T$, $T = \begin{pmatrix} T_1 & & \\ & \ddots & \\ & & T_s \end{pmatrix}$ T_j 1×1 2×2

just evals $\Rightarrow$ only need T

⑥ can get evecs from real Schur form $O(n^3)$

⑦ $O(n^3)$ pre-process, $O(n^2)$ iteration

locally quadratic convergence

can get global convergence

practice: a few iterations (1-3) before deflating

$O(n^3)$

⑧ stability $\Rightarrow$ multiplying by orthog matrices

Symmetric eval problems

$A = A^T$ (or $A = A^H$)

all real evals / evecs!

$$x^H A x = x^H A^H x = \overline{x^H A x}$$

$$\lambda \|x\|_2^2 \qquad\qquad \bar{\lambda} \|x\|_2^2$$

$\lambda = \bar{\lambda}$, λ real

$$Ax = \lambda x$$

$\Rightarrow A = QTQ^T$ Schur form

$$A = A^T \Rightarrow T = T^T \Rightarrow T = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$A = Q \Lambda Q^T$$

$$Av_1 = \lambda_1 v_1 \qquad Av_2 = \lambda_2 v_2$$

$$v_1^T A v_2 = \lambda_2 \, v_1^T v_2$$

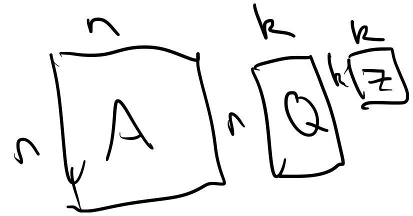
$$v_2^T A v_1 = \lambda_1 \, v_2^T v_1$$

$\lambda_2 \neq \lambda_1$

$v_1^T v_2 = 0$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$AQ = \lambda Q$$



$Z^T Z = I$

$$AQZ = \lambda QZ$$

$$(QZ)^T QZ = Z^T \overbrace{\cancel{Q^T Q}}^{I} Z = I$$