Oct 23, 2020

QR iteration

$T_0 = A$

$Q_k R_k = T_k$

$T_{k+1} = R_k Q_k$

$\Rightarrow$

Shifted QR iteration

$T_0 = A$

σ_k approx eigval

$Q_k R_k = T_k - \sigma_k I$

$T_{k+1} = R_k Q_k + \sigma_k I$

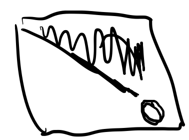
① $R_k = Q_k^T T_k - \sigma_k Q_k^T$

② $Q_k = T_k R_k^{-1} - \sigma_k R_k^{-1}$

$$T_{k+1} = R_k Q_k + \sigma_k I = \left(Q_k^T T_k - \cancel{\sigma_k Q_k^T}\right)\cancel{Q_k} + \cancel{\sigma_k I} = Q_k^T T_k Q_k$$

What if σ_k is evl of T_k? (and of A)

$\Rightarrow T_k - \sigma_k I$ singular $\Rightarrow R_k$ singular



$e_n^T R_k Q_k = 0$

$$\Rightarrow e_n^T T_{k+1} = \sigma_k e_n^T \Rightarrow T_{k+1} = \begin{array}{c} \\ n-1 \\ 1 \end{array}\begin{array}{c} \begin{array}{cc} n-1 & 1 \end{array} \\ \left(\begin{array}{c|c} T_{11} & t_{12} \\ \hline 0 & \sigma_k \end{array}\right) \end{array}$$

Look at T_{11} to get rest of evals

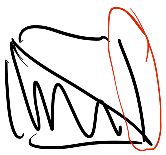
Interpretation as inverse iteration

$Q_k R_k = T_k - \sigma_k I$, $\quad q_n$ last column of Q_k

$$q_n^T (T_k - \sigma_k I) = q_n^T Q_k R_k = e_n^T R_k = 0 \Rightarrow q_n^T T_k = \sigma_k q_n^T$$

$$T_k - \sigma_k I = Q_k R_k \Rightarrow (T_k - \sigma_k I) R_k^{-1} = Q_k$$

$$\Rightarrow R_k (T_k - \sigma_k I)^{-1} = Q_k^T$$



$r_{nn} e_n$

$$\Rightarrow (T_k - \sigma_k I)^{-T} R_k^T e_n = Q_k e_n$$

q_n just comes from one step of II on e_n

Idea: $\sigma_k = T_k(n,n)$ Suppose $\| T_k(n, 1{:}n-1) \| = O(u)$

Consider $\tilde{T}_k$ by setting $T_k(n, 1{:}n-1) = 0 \Rightarrow$ eval equal to $\tilde{\lambda} = T_k(n,n)$

Well-conditioned $\Rightarrow |\lambda - \tilde{\lambda}| = O(u)$

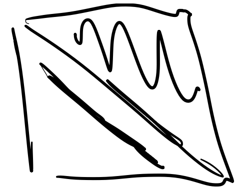
Shift $= \tilde{\lambda} \Rightarrow$ one step of inverse iter. on $T_k - \tilde{\lambda} I$ (e_n)

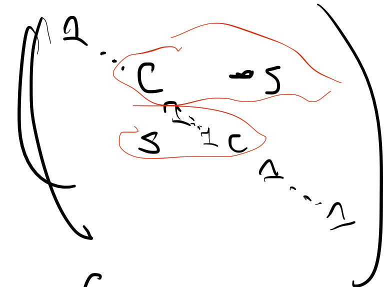
Error $\searrow$ by $\lambda_2[(T_k - \tilde{\lambda} I)^{-1}] \,/\, \lambda_1[(T_k - \tilde{\lambda})^{-1}] = \underbrace{|\lambda - \tilde{\lambda}|}_{O(u)} \cdot \lambda_2[(T_k - \tilde{\lambda} I)^{-1}]$

$O(u) \Rightarrow O(u^2)$

Problem 1: QR fact. at each step is $O(n^3)$

Solution: upper Hessenberg form

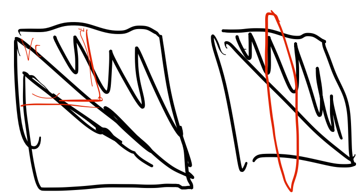
$T_0 = U A U^T = H$ 



$Q_0 R_0 = \overbrace{T_0 - \sigma_0 I}^{H_0}$ $O(n^2)$ time with $O(n)$ Givens

$T_1 = \underbrace{R_0 Q_0}_{H_1} + \sigma_k I$ $O(n^2)$ time (apply Givens rotations)

Claim: T_1 is upper Hess (H_1 is UH)

Proof: $Q_0 = H_0 R_0^{-1} =$ 

R singular $\quad Q_0 \begin{array}{cc} r & n-r \\ \left(\begin{matrix} R_{11} & R_{12} \\ 0 & R_{22} \end{matrix}\right) \end{array}$

UH-ness of T_k is maintained

$O(n^2)$ iteration

Problem 2: all real!

Input: $A \in \mathbb{R}^{n\times n}$

$T_0 = U^T A U' = H$

$\sigma_k = T_k(n,n)$

$Q_k R_k = T_k - \sigma_k I$

$T_{k+1} = R_k Q_k + \sigma_k I$

if $T_{k+1} \approx \left(\begin{array}{c|c} T_{11} & t_{12} \\ \hline \cancel{0^T} \times & \sigma \end{array}\right)$

extract σ_1

continue on with T_{11}

What if λ complex?

Real Schur form

Given any $A \in \mathbb{R}^{n\times n}$

$A = Q T Q^T$, $Q^T Q = Q Q^T = I$

$T = \begin{pmatrix} T_1 & \cdots & \cdots \\ & T_2 & \cdots \\ 0 & & T_s \end{pmatrix}$ T_j 1×1 or 2×2

2×2 blocks correspond to conjugate pairs of complex eigenvalues

$\det\left[\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\right] = (a-\lambda)(d-\lambda) - bc$

Double shifts

Bottom 2×2 block

$\sigma_k, \quad \bar{\sigma}_k$

$$Q_k R_k = T_k - \sigma_k I$$

$$T_{k+1} = R_k Q_k + \sigma_k I$$

$$Q_{k+1} R_{k+1} = T_{k+1} - \bar{\sigma}_k I$$

$$T_{k+2} = R_{k+1} Q_{k+1} + \bar{\sigma}_k I$$

Can show:

$$T_{k+2} = RQ$$

$$QR = (T_k - \sigma_k I)(T_k + \bar{\sigma}_k I)$$

$$= T_k^2 + |\sigma_k|^2 I - 2\,\mathrm{Real}(\sigma_k) T_k$$