Oct 21, 2020

Power method

$$z = A x_k$$

$$x_{k+1} = z / \|z_k\|_2$$

$$\tilde{\lambda}_{k+1} = x_{k+1}^H A x_{k+1}$$

$\lambda_1 \approx \lambda_2$, $|\lambda_2| > |\lambda_3| > \dots \geq |\lambda_n|$

PM: slow! $|\lambda_1/\lambda_2| \approx 1$

Inverse iteration

σ close to λ_1

$\Rightarrow$ close to λ_2

Idea: look for subspace V corresponding to λ_1, λ_2

$$V = E_{\lambda_1} \cup E_{\lambda_2}$$

V invariant subspace

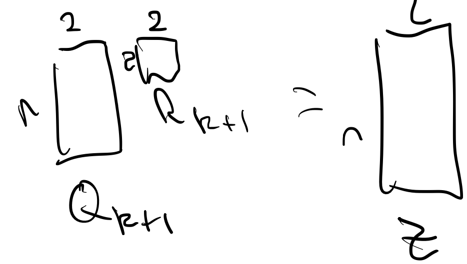
$$AV \subseteq V$$

Problem: need basis for V

Idea: Q_0 $n \times 2$ $Q_0^H Q_0 = I$

$$z = A Q_k$$

$$Q_{k+1} R_{k+1} = z$$



More generally ... orthogonal iteration

Let $1 \leq p < n$ $Q_0^H Q_0 = I$, Q_0 $n \times p$

$$\left.\begin{aligned} z &= A Q_k \\ Q_{k+1} R_{k+1} &= Q_k \end{aligned}\right\} \text{$p = 1$ is just the PM}$$

Evals?

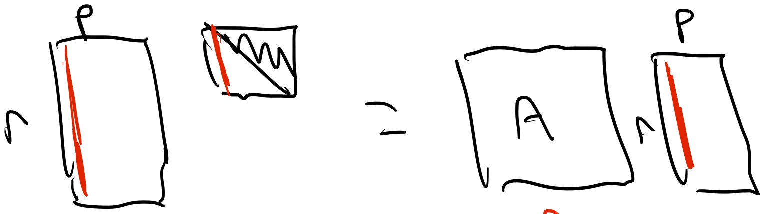
$$\tilde{\Lambda} = \operatorname{diag}(Q_{k+1}^H A Q_{k+1})$$

if q column of Q_{k+1}, $Aq \approx \lambda q$

$$q^H A q \approx \lambda$$

$$\text{eigensolver}(\underbrace{Q_{k+1}^H A Q_{k+1}}_{}^{p \times p})$$

$$Q_{k+1} R_{k+1} = A Q_k$$



$$q_{k+1,1} r_{11} = A q_{k,1}$$

More generally,

$$Q_{k+1}(:, 1:j) R_{k+1}(1:j, 1:j)$$

$$= A Q_k(:, 1:j)$$

nested set of orthog iterations $1 \le j \le p$

$$\to S \begin{pmatrix} \tilde{M}_k \\ 0 \end{pmatrix}$$

$\to$ first p columns of S

Convergence (sketch)

$$A = S \Lambda S^{-1} \qquad |\lambda_p| > |\lambda_{p+1}|$$

Reduced / thin econ QR

$$Z_k = A Q_k \qquad Q_0 = S X_0$$

$$Q_{k+1} R_{k+1} = Z_k$$

$$\mathrm{span}(Q_{k+1}) = \mathrm{span}(Z_k) = \mathrm{span}(A Q_k)$$

$$A Q_k R_k = A A \cdot Q_{k-1} \Rightarrow = \mathrm{span}(A^2 Q_{k-1})$$

$$\mathrm{span}(Q_{k+1}) = \mathrm{span}(A^k Q_0)$$

$$A^k Q_0 = S \Lambda^k S^{-1} S X_0$$

$$= S (\lambda_p I)^k \begin{pmatrix} (\lambda_1/\lambda_p)^k & & \\ & \ddots & \\ & & (\lambda_n/\lambda_p)^k \end{pmatrix} X_0$$

$$= S (\lambda_p I)^k \begin{array}{c} \\ p \\ n-p \end{array} \begin{array}{c} p \\ \begin{pmatrix} M_k \\ N_k \end{pmatrix} \end{array}$$

$M_k \to \ne 0$: λ_m/λ_p, $m < p$

$N_k \to 0$: $\dfrac{\lambda_m}{\lambda_p}$, $m > p$

What happens if $p = n$

At convergence ... $Q_{k+1} R_{k+1} = A Q_k \qquad Q_{k+1} = Q_k$

$\Rightarrow A = Q_k R_{k+1} Q_k^H = Q T Q^H$ (Schur form)

① $Q_k^H A Q_k \to T$

② Diagonals of T are $\lambda_1, \ldots, \lambda_n$

③ use inverse iteration with these eigenvalue estimates?

$$T_k = Q_k^H A Q_k = \underbrace{Q_k^H Q_{k+1}}_{Q} \underbrace{R_{k+1}}_{R}$$

$$T_{k+1} = Q_{k+1}^H A Q_{k+1}$$

$$= Q_{k+1}^H A Q_k Q_k^H Q_{k+1}$$

$Q_{k+1} R_{k+1} = A Q_k \Rightarrow R_{k+1} = Q_{k+1}^H A Q_k$

$$= \underbrace{R_{k+1}}_{R} \underbrace{Q_k^H Q_{k+1}}_{Q}$$

Orthog iteration

$$\underline{Q}_0 = I$$

$$Z_k = A\underline{Q}_k$$

$$\underline{Q}_{k+1} R_{k+1} = Z_k \quad \text{(QR)}$$

$$T_{k+1} = \underline{Q}_{k+1}^T A \underline{Q}_{k+1} \quad \text{(mat mul)}$$

QR iteration

$$T_0 = A$$

$$Q_{k+1} R_{k+1} = T_k \quad \text{(QR)}$$

$$T_{k+1} = R_{k+1} Q_{k+1} \quad \text{(mat mul)}$$

$$\underline{Q}_{k+1} = Q_1 \cdots Q_k$$

Define $\underline{R}_k = R_k \cdots R_1$ for both algorithms (T&B) Ch. 28

Claim: $\underline{R}_k, \underline{Q}_k, T_k$ same, and $A^k = \underline{Q}_k \underline{R}_k$

Proof: (OI) $A^{k+1} = A\underline{Q}_k\underline{R}_k = Z_k\underline{R}_k = \underline{Q}_{k+1} R_{k+1} \underline{R}_k = \underline{Q}_{k+1}\underline{R}_{k+1}$

(QRI) $A^{k+1} = A\underline{Q}_k\underline{R}_k = \underline{Q}_k T_k \underline{R}_k = \underline{Q}_k Q_{k+1} R_{k+1} \underline{R}_k = \underline{Q}_{k+1}\underline{R}_{k+1}$

(OI) $T_{k+1} = \underline{Q}_{k+1}^T A \underline{Q}_{k+1}$

(QRI) $T_{k+1} = R_{k+1} Q_{k+1} = Q_{k+1}^T T_k Q_{k+1} = Q_{k+1}^T \underline{Q}_k^T A \underline{Q}_k Q_{k+1}$

$$= \underline{Q}_{k+1}^T A \underline{Q}_{k+1}$$

Problems:

1. QR alg at every step $O(n^3 \cdot \#\text{iterations})$
2. Everything was real? but λ can be complex?
3. shifts?