Oct 19, 2020 Want $x \neq 0$, λ $Ax = \lambda x$ or $\{(x_i, \lambda_i)\}$

Power iteration

$$y_{k+1} = A x_k$$

$$x_{k+1} = y_{k+1} / \|y_{k+1}\|_2$$

$$\tilde{\lambda}_{k+1} = x_{k+1}^H A x_{k+1}$$

Assume $A = S \Lambda S^{-1}$

$|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$

$$x_{k+1} = A^k x_0 / \|A^k x_0\|_2$$

$$A^k = (S\Lambda \cancel{S^{-1}})(\cancel{S}\Lambda S^{-1}) \cdots (S\Lambda S^{-1})$$

$$= S \Lambda^k S^{-1}, \quad x_0 = S z$$

$$A^k x_0 = S \Lambda^k S^{-1} S z = S \begin{pmatrix} \lambda_1^k z_1 \\ \vdots \\ \lambda_n^k z_n \end{pmatrix}$$

$$= \lambda_1^k z_1 \cdot S \begin{pmatrix} 1 \\ (\lambda_2/\lambda_1)^k z_2/z_1 \\ \vdots \\ (\lambda_n/\lambda_1)^k z_n/z_1 \end{pmatrix}$$

$\Rightarrow S e_1$

= first eigenvector

Problems

1. $z_1 = 0$? random x_0
2. $|\lambda_2| = |\lambda_1|$
3. $|\lambda_2/\lambda_1| \approx 1$
4. only one eigenpair

sym: $A = V \Lambda V^T$

(v_1, λ_1)

$(I - v_1 v_1^T) A (I - v_1 v_1^T)$

Example: Markov chains

$P_{ij} = \text{Prob}(X_{t+1} = i \mid X_t = j)$

column stochastic

P "nice" $\Rightarrow$

$\lambda_1 = 1, \quad Pv = v, \quad v \geq 0$

$|\lambda_1| > |\lambda_2|$

v = stationary distribution

$$y_i = (Pv)_i = \sum_j P_{ij} v_j$$
$$= \Pr(X_{t+1} = i \mid X_t = j) \Pr(X_t = j)$$
$$= \Pr(\text{at state } i)$$

$x_0 \geq 0 \quad \|x_0\|_1 = 1$ starting distribution

$$(Px_0)_i = \sum_j \Pr(X_1 = i \mid X_0 = j) \Pr(X_0 = j)$$
$$= \Pr(X_1 = i \mid x_0)$$

$$(P^k x_0)_i = \Pr(X_k = i \mid x_0)$$

$P = S \Lambda S^{-1} \qquad x_0 = Sz \qquad s_1 = v$

$$P^k x_0 = S \Lambda^k z$$
$$= S \begin{pmatrix} z_1 \\ \lambda_2^k z_2 \\ \vdots \\ \lambda_n^k z_n \end{pmatrix}$$

$z_1 = 1$

$Se_1 = v$

$$P^k x_0 - v = S \begin{pmatrix} 0 \\ z_2 \lambda_2^k \\ \vdots \\ z_n \lambda_n^k \end{pmatrix}$$
$$= S \begin{pmatrix} 0 & & & \\ & \lambda_2^k & & \\ & & \ddots & \\ & & & \lambda_n^k \end{pmatrix} S^{-1}(z - v)$$

$$\|P^k x_0 - v\|_2 \leq \|S\|_2 \|S^{-1}\|_2 \, |\lambda_2|^k \cdot \|z - v\|_2$$

$|\lambda_2| < 1$

Inverse iteration

$A = S\Lambda S^{-1} \Rightarrow (A-\sigma I)^{-1} = S(\Lambda - \sigma I)^{-1} S^{-1}$

Spectral mapping thm: $f(z)$ $f(A) = S f(\Lambda) S^{-1}$ $f(z) = (z-\sigma)^{-1}$

Largest eval of $(A-\sigma I)^{-1}$ is $\max_j \left| \frac{1}{\lambda_j - \sigma} \right|$ $\Rightarrow$ evec closest to λ_j

Idea 1: good guess of some λ_j, all set!

Idea 2: $A\hat{x} - \hat{\lambda}\hat{x} \approx 0$

choose $\sigma = \hat{\lambda} = \dfrac{\hat{x}^H A \hat{x}}{\hat{x}^H \hat{x}} = R_A(\hat{x})$

Rayleigh Quotient iteration

1. $\tilde{\lambda}_{k+1} = \dfrac{x_{k+1}^H A x_{k+1}}{x_{k+1}^H x_{k+1}}$
2. Power iteration on $(A - \tilde{\lambda}_{k+1})^{-1} x_{k+1} \Leftrightarrow$ Solve $(A - \tilde{\lambda}_{k+1}) y = x_{k+1}$
3. $x_{k+2} = y / \|y\|_2$

Problem: what does it converges to? Other eigenvectors?

Hessenberg form

Can we pre-process A in some way?

Last time: $Q^H A Q = T$ Schur form

$$A = \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \xrightarrow{Q_1^H = I - vv^T} \begin{pmatrix} \times & \times & \times & \times \\ \cancel{0} & \times & \times & \times \\ \cancel{0} & \times & \times & \times \\ \cancel{0} & \times & \times & \times \end{pmatrix} \xleftarrow{Q_1 = I - vv^T}$$

Can't expect direct algorithm!

$$A = \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \\ \times & \times & \times & \times \end{pmatrix} \xrightarrow{Q_1^H = \begin{pmatrix} 1 & 0 \\ 0 & I - vv^T \end{pmatrix}} \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & \times & \times & \times \end{pmatrix} \xleftarrow{Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & I - vv^T \end{pmatrix}}$$

$$Q_2^H Q_1^H A Q_1 = \begin{pmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{pmatrix} \xleftarrow{Q_2 = \begin{pmatrix} I & 0 \\ 0 & I - vv^T \end{pmatrix}}$$

$$Q^H A Q = \underbrace{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}_{H}$$

upper Hessenberg form

$H_{ij} = 0$ if $i \geq j+1$ $\qquad O(n^3)$

A, H similar

$$(H - \sigma I)\, y = x$$

Idea: QR: $\tilde{H} = H - \lambda I = QR$

$$R y = Q^T x \qquad O(n^2)$$

G $\begin{pmatrix} x & x & x & x \\ x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{pmatrix}$ $O(n)$ rotations

$O(n)$ update