10/16/2020

① $\underset{n\times n}{A}\,\underset{n\times 1}{x} = \underset{n\times 1}{b}$

② $\underset{x}{\arg\min} \left\| \underset{m\times n}{A}\,\underset{n\times 1}{x} - \underset{m\times 1}{b} \right\|_2$

$$A^T A \hat{x} = A^T b$$

③ $Ax = \lambda x \quad x \neq 0 \qquad (A - \lambda I)x = 0$ eigenpair $(x, \lambda)$ $\quad (\alpha x, \lambda)$

$p(\lambda) = \det(A - \lambda I)$ characteristic polynomial

$\quad = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$

Consequences:

- exactly $n$ eigenvalues (with algebraic multiplicities)
- even if $A \in \mathbb{R}^{n\times n}$, $\lambda_j$ might be complex $\lambda_j \in \mathbb{C}$
- $A \in \mathbb{R}^{n\times n}$, $\lambda_j$ is eval, $\bar{\lambda}_j$ is eval
- if $n \geq 5$, can't have exact algorithms

# Similarity transforms

$B = S^{-1} A S$, $A, B, S$ $n \times n$

Claim: $A$, $B$ same evals

$$B \quad y \quad = \lambda y$$

Proof: $Ax = \lambda x \iff S^{-1} A x = \lambda S^{-1} x \iff S^{-1} A S S^{-1} x = \lambda S^{-1} x$

Theme: transform $A$ to $B$, where evals easier to compute

$By = \lambda y \Rightarrow Ax = \lambda x$, $x = Sy$

First idea: $B$ diagonal $\Rightarrow Be_i = B_{ii} e_i \Rightarrow Se_i$ evec

Problem: sometimes impossible!

$S$ nonsingular $\Rightarrow$ $n$ LI eigenvectors of $A$

$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $\quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ 2x_2 \end{pmatrix}$ $\quad x_2 \neq 0 \Rightarrow \lambda = 2$

$A e_1 = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 e_1$

only one LI eigenvector

$E_\lambda = \{ x \mid Ax = \lambda x \}$ invariant subspace : $A E_\lambda \subseteq E_\lambda$

$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ $\dim(E_2) = 1 =$ geometric multiplicity

$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^2$

$\det\left(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \lambda I\right) = (2-\lambda)^2$

Jordan form For any $A$, $J = S^{-1} A S = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}$

$J_i$ is Jordan block, $J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$ $(\lambda_i)$

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Everything is much simpler if $A = A^T$ $(AA^T = A^T A)$ normal

$A = V \Lambda V^T$ $V^T A V = \Lambda$

Everything is real

Jordan form unstable

$Q^H A Q = T \qquad M_{ij}^H = \overline{M_{ji}}$

Schur factorization: $A = QTQ^H$

Evals of $T$ on diagonal:

$\det(T - \lambda I) = \prod_{i=1}^{n} (t_{ii} - \lambda)$

$Tx = t_{ii} x$

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ 0 & t_{ii} & T_{23} \\ 0 & 0 & T_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t_{ii} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$T_{33} x_3 = t_{ii} x_3$

$T_{nn} \neq t_{ii} \Rightarrow x_3 = 0$

$t_{ii} x_2 = t_{ii} x_2 \Rightarrow x_2 = 1$

$T_{11} x_1 + T_{12} = t_{ii} x_1$

$(T_{11} - t_{ii} I) x_1 = -T_{12}$

Let $Ax = \lambda x \qquad \|x\|_2 = 1$

$V = [x \;\; Q_1] \qquad V^H V = V V^H = I$

$$V^H A V = \begin{pmatrix} \cancel{\lambda x^T x}\ 1 & \overbrace{x^T \cancel{A} Q_1}^{B} \\ \underbrace{\cancel{Q_1^T \lambda x}}_{0} & \underbrace{Q_1^T \cancel{A} Q_1}_{C} \end{pmatrix}$$

$Q_2^H C Q_2 = T_2$

$$V^H \begin{pmatrix} 1 & 0 \\ 0 & Q_2^H \end{pmatrix} A \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & Q_2 \end{pmatrix} V}_{Q}$$

$$= \underbrace{\begin{pmatrix} 1 & B \\ 0 & T_2 \end{pmatrix}}_{T}$$

If $A = A^H$

$QTQ^H = QT^H Q^H$

$\Rightarrow T = T^H$

$\Rightarrow T$ diag.

## Why eigenvalues?

Optimization: $\|A\|_2^2 = \max_{\|x\|_2^2=1} \|Ax\|_2^2 = \max_{\|x\|_2^2=1} x^T A^T A x$

$A^T A = V \Lambda V^T \quad x = Vy \quad \|x\|_2^2 = 1 \iff \|y\|_2^2 = 1 \quad y^T \cancel{V^T} V \Lambda \cancel{V} V^T y$

$\max_{\|y\|_2^2=1} \sum_{i=1}^n y_i^2 \lambda_i \Rightarrow y = e_1 \quad x = Ve_1 = v_1$

$M = M^T \quad \min_{\|x\|_2^2=1} x^T M x \quad M = V \Lambda V^T \quad x = Vy \Rightarrow \sum_{i=1}^n y_i^2 \lambda_i$

$\Rightarrow y = e_n \Rightarrow x = Ve_n = v_n$

Rayleigh Quotient: $R(x) = \dfrac{x^T M x}{x^T x}$

$\{R(x) \mid x \in \mathbb{R}^n\} = \{x^T M x \mid \|x\|_2^2 = 1\}$

$\min/\max \quad x^T M x$

$\text{s.t.} \quad x^T x = 1$

$L(x,\gamma) = x^T M x - \gamma(x^T x - 1)$

$\nabla_x L(x,\gamma) = 2Mx - 2\gamma x = 0 \Rightarrow Mx = \gamma x$

$\frac{\partial}{\partial \gamma} L(x,\gamma) = x^T x - 1 = 0 \Rightarrow \|x\|_2 = 1$