Oct 5, 2020 LLS: $\hat{x} = \arg\min_x \|Ax-b\|_2^2 \quad A = QR \quad \hat{x} = R^{-1}Q^Tb$

LU derivation

$$A = \begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \xrightarrow{L_1 A} \begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{pmatrix} \xrightarrow{L_2 L_1 A} \begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{pmatrix} = U$$

$$L_2 L_1 A = U \Rightarrow A = (L_1^{-1} L_2^{-1}) U = LU$$

Now for QR today!

Q_i are $m \times m$

$$A = \begin{pmatrix} x & x \\ x & x \\ x & x \end{pmatrix} \xrightarrow{Q_1 A} \begin{pmatrix} x & x \\ 0 & x \\ 0 & x \end{pmatrix} \xrightarrow{Q_2 Q_1 A} \begin{pmatrix} x & x \\ 0 & x \\ 0 & 0 \end{pmatrix}$$

$$Q_2 Q_1 A = \begin{pmatrix} R \\ 0 \end{pmatrix} \Rightarrow A = Q_1^T Q_2^T \begin{pmatrix} R \\ 0 \end{pmatrix} = \underbrace{\overbrace{\boxed{Q}}^{m}}_{m} \; {}_{m}\overbrace{\boxed{\begin{matrix} R \\ 0 \end{matrix}}}^{n}$$

Things to worry about:

✓ ① multiply by $m \times m$ matrices. expensive?

✓ ② form Q efficiently?

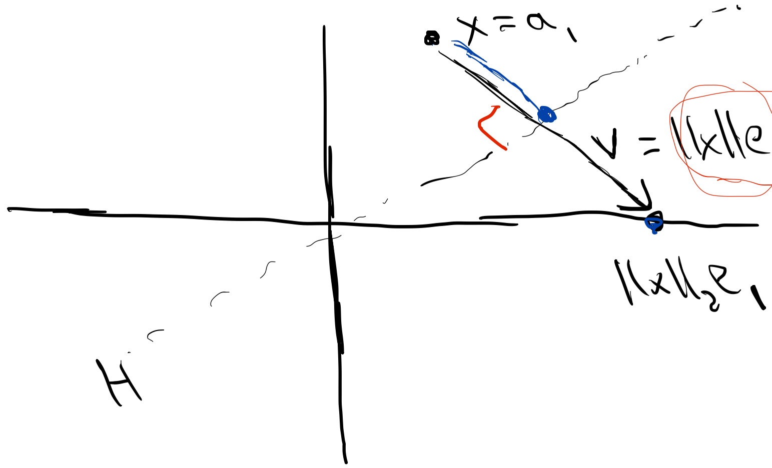
✓ ③ ${}_{m}\overset{n}{\boxed{A}} = {}_{m}\overset{n}{\boxed{Q}} \; {}_{n}\overset{n}{\boxed{R}}$ $\quad Q^TQ = I \quad R$ upper tri

Householder reflectors

$$A = \begin{pmatrix} x & x \\ x & x \\ x & x \end{pmatrix} \xrightarrow{Q_1 A} \begin{pmatrix} x & x \\ 0 & x \\ 0 & x \end{pmatrix}$$

$$Q_1 a_1 = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \alpha e_1$$

$$\|Q_1 a_1\|_2 = \|a_1\|_2 \Rightarrow Q_1 a_1 = \|a_1\|_2 e_1$$



Project onto e_1 orthogonally to v

Ortho project onto v: $vv^T / \|v\|_2^2$

Complementary proj: $I - vv^T / v^Tv$

Ortho proj. onto H: $I - \frac{vv^T}{v^Tv}$ for reflection, need $I - 2\frac{vv^T}{v^Tv}$

first entry of $x = e_1^T x$

$$\left(I - 2\frac{vv^T}{v^Tv}\right) x = x - 2\frac{(\|x\|_2 x_1 - \|x\|_2^2)(\|x\|_2 e_1 - x)}{\|x\|_2^2 + \|x\|_2^2 - 2\|x\|_2 x_1}$$

$$= x - (\|x\|_2 e_1 - x)\frac{2\|x\|_2 x_1 - 2\|x\|_2^2}{2\|x\|_2^2 - 2\|x\|_2 x_1}$$

$$= x - (\|x\|_2 e_1 - x)(-1) = \|x\|_2 e_1$$

$$\left(I - 2\frac{vv^T}{v^Tv}\right)\left(I - 2\frac{vv^T}{v^Tv}\right)$$

$$= I + \cancel{4\frac{v v^T v v^T}{(v^Tv)^2} - 4\frac{vv^T}{v^Tv}}$$

Note: $\|x\|_2 e_1$ or $-\|x\|_2 e_1$ $(v' = -\|x\|_2 e_1 - x)$

$s = \pm 1$ $\quad s\|x\|_2 e_1 - x$ $\quad x_1 > 0$ $s = -1$, $\quad x_1 \leq 0$ $s = +1$

$$\begin{pmatrix} x & x \\ x & x \\ x & x \end{pmatrix} \xrightarrow{Q_1 A} \begin{pmatrix} x & x \\ 0 & x \\ 0 & x \end{pmatrix}$$

$$Q_1 A = \begin{array}{c} \\ m \end{array} \left[\begin{array}{c|c} \begin{matrix} x \\ 0 \\ \vdots \\ 0 \end{matrix} & \overset{n-1}{A_2} \end{array} \right]$$

$$\begin{array}{cc} & \begin{matrix} 1 & \end{matrix} \\ 1 & \begin{pmatrix} 1 & 0 \\ 0 & \bar{Q}_2 \end{pmatrix} \end{array} \left[\begin{matrix} x \\ 0 \\ \vdots \\ 0 \end{matrix} \quad A_2 \right]$$

$\bar{Q}_2$ householder reflector for $A_2(2:m, 1)$

Householder QR

for $j = 1:n$

$\quad x = A(j:m, j)$

$\quad s = -\text{sign}(x_1)$

$\quad v_j = s\|x\|_2 e_1 - x$

$\quad v_j \mathrel{/}= \|v_j\|_2$

$\quad A(j:m, j:n) \mathrel{-}=$

$\qquad 2 v_j v_j^T A(j:m, j:n)$

$$\left(I - \frac{v v^T}{v^T v}\right) A$$

Optimizations

$A(j,j) = s\|x\|_2 = r_{jj}$

$A(j+1:m, j) = 0$ don't store explicitly

can overwrite A with v_j

(need extra entry)

n iterations

$O(mn)$ flops

$O(mn^2)$

LLS: $R\hat{x} = Q^T b$

$A = QR$

$Q_n \cdots Q_1 A = R$

$Q^T A = Q^T Q R = R$

$Q^T b = Q_n \cdots Q_1 b$

$$Q_j = \begin{array}{c c} & \begin{array}{cc} j-1 & m-j+1 \end{array} \\ \begin{array}{c} j-1 \\ m-j+1 \end{array} & \begin{pmatrix} I & O \\ O & I - 2 v_j v_j^T \end{pmatrix} \end{array}$$

Update with Q_j

$$b(j:m) \mathrel{-}= 2 v_j \left(v_j^T b(j:m) \right)$$

Flops: $\sum_{j=1}^{n} c(m-j+1)$

$= O(mn)$

no need to form Q!

Note: $Q_i = Q_i^T$

$Qb = Q_1 \cdots Q_n b$

can do same updating

<u>If we want Q?</u>

$Qe_1 =$ first column of Q

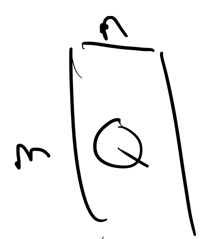
$Q_1 \cdots Q_n e_1 = Q_1 e_1$

$Q_1 \cdots Q_n e_2 = Q_1 Q_2 e_2$

$\vdots$

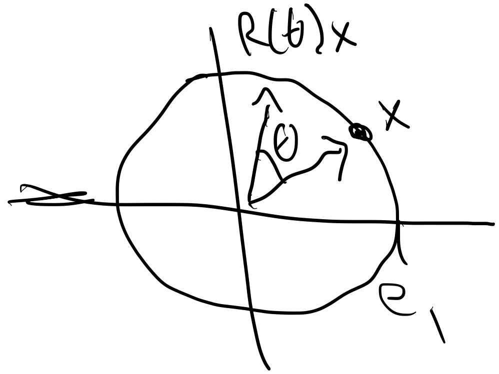
$Q_1 \cdots Q_n e_n$

$O(mn^2)$



Givens rotations

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



Idea: rotate to $\|x\|_2 e_1$

$$\cos\theta = \frac{x_1}{\sqrt{x_1^2+x_2^2}} \qquad \sin\theta = \frac{-x_2}{\sqrt{x_1^2+x_2^2}}$$

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt{x_1^2+x_2^2} \\ 0 \end{pmatrix}$$

$$G = \begin{array}{c} \\ i \\ j \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccccc} & & i & & j & & \end{array} \\ \begin{pmatrix} 1 & \cdots & & & & & \\ & 1 & & & & & \\ & & c & 1 & -s & & \\ & & & \ddots & & & \\ & & s & & c & & \\ & & & & & 1 & \cdots \\ & & & & & & 1 \end{pmatrix} \end{array}$$

$A_{ik},\ A_{jk} \Rightarrow G$

$(GA)_{jk} = 0 \qquad (GA)_{ik} \neq 0$

if $A_{ik'},\ A_{jk'} = 0$

$(GA)_{ik'} = (GA)_{jk'} = 0$

if one is non-zero, can both fill in

$$
\begin{pmatrix}
X & X & X & X & X \\
O & X & X & X & X \\
O & O & X & X & X \\
O & O & X & X & X \\
O & O & X & O & X
\end{pmatrix}
$$