Oct 2, 2020

Last time: linear LS: $\hat{x} = \arg\min_x \|Ax-b\|_2^2$

$$\hat{x} = A^+ b = (A^TA)^{-1}A^T b = V\Sigma^{-1}U^T b = \underline{R^{-1}Q^T b}$$

$$\underset{m \times n}{A} = \underset{m \times n}{Q}\ \underset{n \times n}{R} \qquad Q^TQ = I$$

$$\begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix} = \begin{pmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} r_{11} & \cdots & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ & & \ddots & \vdots \\ 0 & & & r_{nn} \end{pmatrix}$$

$$a_1 = q_1 r_{11} \longrightarrow q_1 = \frac{a_1}{\|a\|_2} \qquad q_1^T q_1 = \frac{a_1^T a_1}{\|a\|_2^2} = 1$$

$$a_2 = q_1 r_{12} + q_2 r_{22} \qquad r_{11} = \|a\|_2$$

$$q_1^T a_2 = \underbrace{q_1^T q_1}_{1} r_{12} + \underbrace{\cancel{q_1^T q_2}}_{0} r_{22} \qquad r_{12} = q_1^T a_2$$

$$\underbrace{a_2 - q_1 r_{12}}_{v_2} = q_2 r_{22} \qquad q_2 = \frac{v_2}{\|v_2\|_2} \qquad r_{22} = \|v_2\|_2$$

$$q_i^T a_j = \underbrace{q_i^T q_i}_{1} r_{ij} + \cdots + \underbrace{\cancel{q_i^T q_j}}_{0} r_{jj}$$

$$\underline{r_{ij} = q_i^T a_j}$$

(Classical) Gram-Schmidt

for $j = 1:n$

$\quad v_j = a_j$

$\quad$ for $i = 1:j-1$

$\quad\quad r_{ij} = q_i^T a_j$ $\quad O(m)$

$\quad\quad v_j -= r_{ij} q_i$

$\quad r_{jj} = \|v_j\|_2$

$\quad q_j = v_j / r_{jj}$

$r_{jj} = 0$?

r_{jj} close to 0?

$O(n)$ iterations

$O(m \cdot n^2)$ total

$$\begin{bmatrix} a_1 & A_2 \end{bmatrix} = \begin{bmatrix} q_1 & Q_2 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12}^T \\ 0 & R_{22} \end{bmatrix}$$

$a_1 = q_1 r_{11} \quad q_1 = \frac{a_1}{\|a_1\|_2} \quad r_{11} = \|a_1\|_2$

$$q_1^T A_2 = \underbrace{q_1^T q_1}_{1} r_{12}^T + \underbrace{q_1^T Q_2}_{0} R_{22}$$

$$r_{12}^T = q_1^T A_2$$

$$A_2 - q_1 r_{12}^T = Q_2 R_{22}$$

Modified Gram-Schmidt

for $j = 1:n$

$\quad r_{jj} = \|A(:,j)\|_2$

$\quad A(:,j) = A(:,j)/r_{jj}$

$\quad r(j+1:n) = A(:,j)^T A(:,j+1:n)$

$\quad A(:,j+1:n) -= q_i^T A(:,j+1:n)$

$Q = A$

$r_{ji} = q_j^T a_i$

$\quad = q_j^T \left(a_i - \sum_{\ell=1}^{j-1} q_\ell r_{\ell i}\right)$

Q is just an orthonormal basis for range(A)

$Q^TQ = I \qquad QQ^T = ?$

$(QQ^T)^2 = QQ^TQQ^T = QQ^T$

Example of a projector: $P^2 = P$

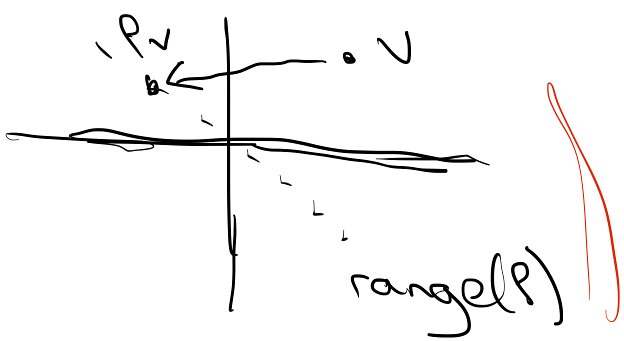
$y \in \text{Range}(P)$

$y = Px \Rightarrow Py = P^2x = Px = y$

$P(Pv - v) = P^2v - Pv$
$= Pv - Pv = 0$

$\Rightarrow Pv - v \in \text{Null}(P)$

$(I-P)v \in \text{Null}(P)$



$(I-P)^2 = I - 2P + P^2$
$= I - 2P + P = I - P$

projects onto null(P)

range$(I-P) \subseteq$ null(P)

if $Pz = 0 \quad (I-P)z = z - Pz = z$

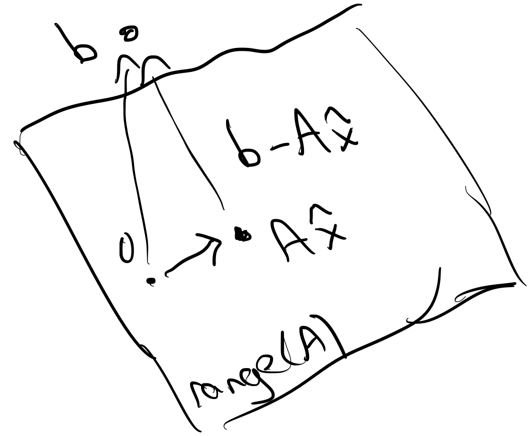
$\Rightarrow z \in$ range$(I-P)$

range$(I-P) =$ null(P)

$$\hat{x} = \arg\min_x \|Ax-b\|_2^2$$

$$\hat{x} = A^+ b$$

$$A \cdot A^+ = QRR^{-1}Q^T = QQ^T$$



QQ^T is an orthogonal projector: it projects onto a subspace along an orthogonal subspace

"projects onto" $\Rightarrow$ range(P)

"along" $\Rightarrow$ null(P)

range(P) $\perp$ null(P)

$$A^T(b - A\hat{x}) = 0 \qquad A\hat{x}$$

project onto range(A) along null(A^T)

(also spectral projectors)

Theorem: P is an orthog. proj iff $P = VV^T$ $\quad V^TV = I$

$(\Rightarrow)$ $P = VV^T$ $\quad P = P^T$ $\quad \text{null}(P) \perp \text{range}(P^T) = \text{range}(P)$ ✓ $\quad P^2 = P$ ✓

$(\Leftarrow)$ P orthog. proj. $\Rightarrow$ $\text{null}(P) \perp \text{range}(P)$

$V_1 =$ ortho basis for $\text{range}(P)$

$V_2 =$ $\qquad \text{null}(P)$

$\bar{V} = [V_1 \; V_2]$ is ortho basis for $\mathbb{R}^m$

$$P\bar{V} = [PV_1 \; PV_2] = [V_1 \; 0]$$

$$\bar{V}^T P \bar{V} = \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} [V_1 \; 0] = \begin{bmatrix} V_1^T V_1 & 0 \\ V_2^T V_1 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

$$P = [V_1 \; V_2] \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$$

$$= [V_1 \; V_2] \begin{bmatrix} V_1^T \\ 0 \end{bmatrix} = V_1 V_1^T \quad ✓$$

CGS

$$v_j = a_j - \sum_{i=1}^{j-1} \underbrace{q_i^T a_j}_{r_{ij}} q_i$$

$$Q_{j-1} = [q_1 \cdots q_{j-1}]$$

$$v_j = \underbrace{(I - Q_{j-1}Q_{j-1}^T)}_{\text{projects onto null}(Q_{j-1}^T)} a_j$$

$$q_j = v_j / \|v_j\|_2$$

$$r_{jj} = \|v_j\|_2$$

MGS

$$[a_1 \quad A_2]$$

$$q_1 = a_1 / \|a_1\|_2$$

$$A_2 = \underbrace{(I - q_1 q_1^T)}_{\text{project onto null}(q_1^T)} A_2$$

$$v_j = \underbrace{(I - q_{j-1}q_{j-1}^T) \cdots (I - q_1 q_1^T)}_{= (I - Q_{j-1}Q_{j-1}^T)} A(:,j)$$