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Symmetric positive definite (SPD)

$A = A^T \qquad x^T A x > 0 \quad \forall x \in \mathbb{R}^n \qquad A = V \Lambda V^T \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad \lambda_i > 0$

Cholesky factorization: $A = LL^T$ (no pivoting)

$$A = \begin{pmatrix} a_{11} & a_{21}^T \\ a_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 \\ l_{21} & L_{22} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21}^T \\ 0 & L_{22}^T \end{pmatrix}$$

$a_{11} = l_{11}^2 \Rightarrow l_{11} = \sqrt{a_{11}}$

$a_{21} = l_{21} l_{11} \Rightarrow l_{21} = a_{21}/\sqrt{a_{11}}$

$A_{22} = L_{22} L_{22}^T + l_{21} l_{21}^T$

$$\underbrace{A_{22} - l_{21} l_{21}^T}_{= S} = L_{22} L_{22}^T$$

$(Px)^T$

$\shortparallel$

$(x^T P^T) M (Px)$

✓ Claim: $a_{11} > 0$

Proof: $e_1^T A e_1 = a_{11} > 0$

Claim: S is SPD

① A SPD $\Leftrightarrow$ A^{-1} SPD

② M SPD $\Leftrightarrow$ any block SPD

$$\begin{pmatrix} x_B^T & 0 \end{pmatrix} \begin{pmatrix} M_{BB} & m \\ m & \nu \end{pmatrix} \begin{pmatrix} x_B \\ 0 \end{pmatrix} = x_B^T M_{BB} x_B > 0$$

③ $S^{-1} = (A^{-1})_{(2,2)}$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22}\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}$$

$$A_{11}B_{12} + A_{12}B_{22} = 0$$

$$B_{12} = -A_{11}^{-1}A_{12}B_{22}$$

$$A_{21}B_{12} + A_{22}B_{22} = I$$

$$-A_{21}A_{11}^{-1}A_{12}B_{22} + A_{22}B_{22} = I$$

$$\underbrace{(A_{22} - A_{21}A_{11}^{-1}A_{12})}_{=S}\underbrace{B_{22}}_{S^{-1}} = I$$

A SPD $\Rightarrow$ A^{-1} SPD

$A^{-1} = B \Rightarrow B_{22} = S^{-1}$ SPD

$\Rightarrow S$ SPD

Do we need pivoting? No

Cholesky

$$(A + \delta A)\hat{x} = b \qquad \frac{\|\delta A\|}{\|A\|} = O(\varepsilon)$$

Some intuition

① $A = LL^T \qquad L = U\Sigma V^T$

$$A = U\Sigma V^T V \Sigma U^T = U\Sigma^2 U^T$$

$$\|A\|_2 = \sigma_1^2 \qquad \|L\|_2 = \sigma_1$$

$$\|L\|_2 \|L^T\|_2 = \|A\|_2$$

② $\| |\hat{L}| \| \; \| |\hat{U}| \|$

$$\|L\|_\infty \le \sqrt{n}\,\|L\|_2$$

$$\| |L| \|_\infty = \|L\|_\infty$$

③ $\hat{L}$ is close to L

Algorithm: $A = LL^T$ (Cholesky)

for $j = 1 : n-1$

$L(j,j) = \sqrt{A(j,j)}$

$L(j+1:n, j) = A(j+1:n, j) / L(j,j)$

$A(j+1:n, j+1:n) \mathrel{-}= L(j+1:n, j)\, \overbrace{L(j, j+1:n)}^{= L(j+1:n,\, j)^T}$

Fact: $A = LU$ w/o pivoting

diagonal D, $D_{jj} = \sqrt{u_{jj}} \Rightarrow A = (LD^{1/2})(D^{-1/2}U)$

$= L_c L_c^T$

Observation: only need access to lower triangular part of A

$\Rightarrow$ save 1/2 flops

1/2 storage

Symmetric indefinite

$$A = LU = LDD^{-1}U = LDM^T$$

D = diag of U $\qquad$ M^T = unit upper tri

$$A^T = MDL^T = M(LD)^T = M\tilde{U} = LU = A$$

uniqueness of $LU \Rightarrow M = L \Rightarrow A = LDL^T$

1/2 flops 1/2 storage

Bunch-Kaufman

$$PAP^T = LDL^T$$

Diagonally dominant $\quad |A_{jj}| \geq \sum_{j \neq i} |A_{ij}|$

Claim: no need to pivot with GEPP

first step: $|A_{11}| \geq \sum_{j \neq i} |A_{i1}| \Rightarrow \underset{l \geq 1}{\arg\max} |A_{i1}| = 1$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l_{21} & L_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ 0 & U_{22} \end{pmatrix}$$

$S = A_{22} - a_{21} a_{12} / a_{11}$ $\quad$ Claim: S is DD

Fix j: $\sum_{i=1, i \neq j}^{n-1} |S_{ij}| = \sum_{i=1, i \neq j}^{n-1} |a_{i+1,j+1} - a_{i+1,1} a_{1,j+1} / a_{11}|$

$\leq \sum_{i=1, i \neq j}^{n-1} |a_{i+1,j+1}| + |a_{i+1,1} a_{1,j+1} / a_{11}|$

$= \sum_{i=1, i \neq j} |a_{i,j+1}| - |a_{1,j+1}| + \sum_{i=1, i \neq j} |a_{i+1,1} a_{1,j+1} / a_{11}| - |a_{j+1,1} a_{1,j+1} / a_{11}|$

$\leq |a_{j+1,j+1}| - |a_{1,j+1}|$

$$\leq |a_{j+1,j+1}| - |a_{1,j+1}| + |a_{1,j+1}| \left(- \left| \frac{a_{j+1,1}}{a_{11}} \right| + \sum_{i=1}^{n-1} \left| \frac{a_{i+1,1}}{a_{11}} \right| \right)$$

$$\leq |a_{j+1,j+1}| - |a_{1,j+1}| \left| \frac{a_{j+1,1}}{a_{11}} \right| \qquad \leq 1$$

$$\leq |a_{j+1,j+1} - a_{1,j+1} a_{j+1,1} / a_{11}|$$

$$= |s_{jj}|$$

"Data sparse"

Toeplitz: $T(i,j) = t(i-j)$ convolutions

Hankel: $H(i,j) = h(i+j)$

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