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GEPP

$L = I, \quad P = I$

for $j = 1:n-1$

  pick $k = \underset{l \geq j}{\operatorname{argmax}} |A(l,j)|$

  swap rows k and j of A, L

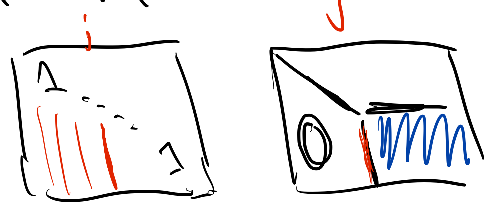
  update P

  $L(j+1:n, j) = A(j+1:n, j) / A(j,j)$

  $A(j+1:n, j) = 0$

  $A(j+1:n, j+1:n) \mathrel{-}= L(j+1:n, j)\, A(j, j+1:n)$

$U = A$



$PA = LU$

$Ax = b \Rightarrow$ ① $c = P^T b$ ② $Ly = c$ ③ $Ux = y$

Error analysis

$$A + E = \hat{L}\hat{U} \qquad |E| \leq n\varepsilon |\hat{L}||\hat{U}|$$

$$\hat{u}_{jk} = a_{jk}$$

step 1: $\hat{u}_{jk} = \hat{u}_{jk} - \hat{l}_{j1}\hat{u}_{1k}$

$\vdots$

step $j-1$: $\hat{u}_{jk} = \hat{u}_{jk} - \hat{l}_{j,j-1}\hat{u}_{j-1,k}$

$$\hat{u}_{jk} = fl\left(a_{jk} - \sum_{i=1}^{j-1} \hat{l}_{ji}\hat{u}_{ik}\right)$$

$$= a_{jk}(1+\delta) - \left(\sum \hat{l}_{ji}\hat{u}_{ik}(1+\delta_i)\right)(1+\delta)$$

$|\delta| \leq \varepsilon, \quad |\delta_i| \leq (j-1)\varepsilon + O(\varepsilon^2)$ HW1

$$a_{jk} = \frac{1}{1+\delta}\hat{u}_{jk}\hat{l}_{jj} + \sum \hat{l}_{ji}\hat{u}_{ik}(1+\delta_i) \qquad \hat{l}_{jj} = 1$$

$\rightarrow 1 - \delta + \delta^2 \ldots = 1 + \delta'$

$|\delta'| \leq \varepsilon + O(\varepsilon^2)$

$$a_{jk} = \sum_{i=1}^{j} \hat{l}_{ji}\hat{u}_{ik} + e_{jk}$$

$$\left.\begin{aligned} |e_{jk}| &\leq \left|\sum_{i=1}^{j} \hat{l}_{ji}\hat{u}_{ik}\delta_i\right| \\ &\leq j\varepsilon(|\hat{L}||\hat{U}|)_{jk} \end{aligned}\right\} \text{HW1}$$

Backward stability of GEPP

$$A + E = \hat{L}\hat{U}, \quad |E| \le n\,\varepsilon\,|\hat{L}|\,|\hat{U}|$$

BW/FW sub $\Rightarrow (A+\delta A)\hat{x} = b \quad |\delta A| \le 3n\,\varepsilon\,|\hat{L}|\,|\hat{U}|$

Backward stable if $\dfrac{\|\delta A\|}{\|A\|} = O(\varepsilon)$

if so, we have $\|\hat{x}-x\|/\|x\| = O(k(A)\,\varepsilon)$

$$\|\delta A\| = O(\varepsilon \cdot \|\,|\hat{L}|\,\| \cdot \|\,|\hat{U}|\,\|) = O(\varepsilon\,\|\hat{L}\|\,\|\hat{U}\|)$$

$$\|M\| = \max_{ij} |M_{ij}|$$

$\|\hat{L}\|_{max} = 1$ (GEPP)

$$\frac{\|\delta A\|}{\|A\|} = O\left(\varepsilon \frac{\|\hat{U}\|}{\|A\|}\right) \qquad \text{growth factor } g = \frac{\max_{ij} |\hat{u}_{ij}|}{\max_{ij} |a_{ij}|}$$

$$\Rightarrow \frac{\|\hat{x}-x\|}{\|x\|} \le k(A) \cdot 3 \cdot n \cdot g \cdot \varepsilon \qquad k(A) = \|A\|\,\|A^{-1}\|$$

How big can g be?

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \qquad \|A\|_{max} = 1$$

first step: no need to pivot

$$L(2{:}4, 1) = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \qquad A(1, 2{:}4) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 2 \\ 0 & -1 & -1 & 2 \end{pmatrix} \quad \ldots \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 8 \end{pmatrix} = U$$

$$\|U\|_{max} = 8 \qquad g = 8/1 = 8$$

more generally, g as large as 2^{n-1} $\quad (g_{PP} \leq 2^{n-1})$

partial solution: complete pivoting $\Rightarrow g \leq n^{\log n/4}$

needs $O(n^3)$ lookups, not used in practice

Structure of error

Want: x s.t. $Ax = b$

Given: $\hat{x}$ s.t. $A\hat{x} \approx b$ $\quad r = A\hat{x} - b$ = residual

$A\hat{x} - Ax = r + b - b = r \Rightarrow A(\hat{x} - x) = r$

$\Rightarrow \hat{x} - x = A^{-1} r$

$\|\hat{x} - x\|_2 = \|A^{-1} r\|_2 \quad A^{-1} = V \Sigma^{-1} U^T$

$= \|\Sigma^{-1} U^T r\|_2 = \|\Sigma^{-1} s\|_2 \quad s = U^T r \quad \|s\|_2 = \|r\|_2$

$$\|\Sigma^{-1} s\|_2^2 = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} s_i^2$$

Bounds $\|\hat{x} - x\| \leq O(\kappa(A)\, \varepsilon \|x\|) = O\left(\frac{\sigma_1}{\sigma_n} \varepsilon \|x\|\right)$

scale so that $\sigma_1 = O(1)$

Worst-case: "mass" of s on n^{th} component

residual $s = U^T r = e_n \Rightarrow r = u_n$

residual in nearly singular subspace

Lecture 5: $(A+\delta A)(x+\delta x) = b$

$$\|\delta x\| = \|A^{-1}\delta A\, x\| \le \|A^{-1}\|\,\|\delta A\|\,\|x\|$$

$$\|\cdot\| = \|\cdot\|_2$$

$$\|A^{-1}\delta A\, x\|_2^2 = \|\Sigma^{-1}U^T \delta A\, x\|_2^2$$

$$\delta A = u_n e_n^T \qquad x = e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\|\delta A\|_2^2 = \max_{\|y\|_2^2} \|u_n e_n^T y\|_2^2 = \max_{\|y\|_2^2} y_n^2 \cancel{\|u_n\|_2^2} \le 1$$

$$y_n = 1 \quad (y = e_n)$$

$$\|x\|_2^2 = 1$$

$$\|\Sigma^{-1}\underbrace{U^T u_n}_{e_n}\underbrace{e_n^T e_n}_{1}\|_2 = \|\Sigma^{-1}e_n\|_2 = \frac{1}{\sigma_n}$$

$$\|A^{-1}\delta A\, x\|_2 = \frac{1}{\sigma_n}$$

$$= \|A^{-1}\|_2 \cdot \|\overset{1}{\cancel{\delta A}}\|_2 \cdot \|\overset{1}{\cancel{x}}\|_2$$