Sep 21, 2020

Gaussian elimination

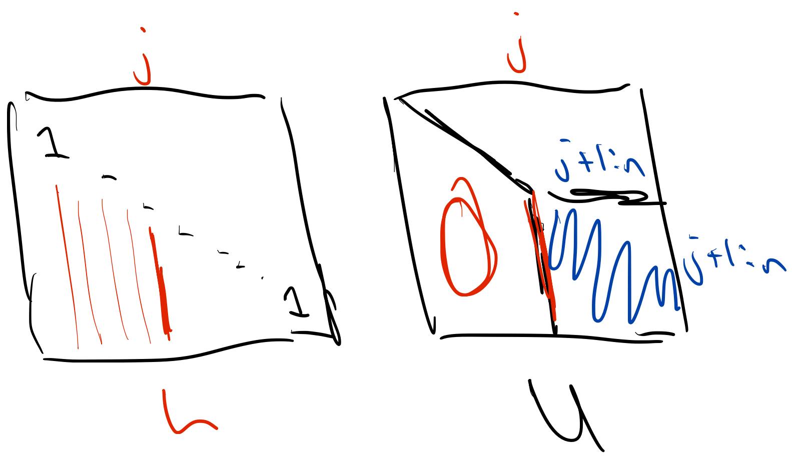
$L = I$

for $j = 1:n-1$

Gauss transf.: $(I + \tau_j e_j^T)A$

- $L(j+1:n, j) = A(j+1:n, j)/A(j,j)$
- $A(j+1:n, j) = 0$
- $A(j+1:n, j+1:n)$ -= $L(j+1:n, j)\,A(j, j+1:n)$

$U = A$



Potential problem: $A(j,j) = 0$

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ Solution: pivoting

A nonsingular $\Rightarrow$ at least one non-zero in the first col

$$PA = \begin{pmatrix} a_{11} & a_{12}^T \\ a_{21} & A_{22} \end{pmatrix}$$

$a_{11} \neq 0$

$$PA = \underbrace{\begin{pmatrix} 1 & 0 \\ a_{21}/a_{11} & I \end{pmatrix}}_{L_1} \underbrace{\begin{pmatrix} a_{11} & a_{12}^T \\ 0 & A_{22} - a_{21}a_{12}^T/a_{11} \end{pmatrix}}_{U_1}$$

$$A_{22} - a_{21}a_{12}^T / a_{11} = S$$

S is the schur complement

$$\underset{\neq 0}{\det(A)} = \det(PA)$$
$$= \underbrace{\det(L_1)}_{1} \det(U_1)$$
$$= \underbrace{a_{11}}_{\neq 0} \cdot \det(S) \Rightarrow \det(S) \neq 0$$

$$PA = \begin{pmatrix} 1 & 0 \\ a_{21}/a_{11} & I \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}^T \\ 0 & S \end{pmatrix} \qquad P_1 S = L_1 U_1 \Rightarrow S = P_1^T L_1 U_1$$

$$= \begin{pmatrix} 1 & 0 \\ a_{21}/a_{11} & P_1^T L_1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}^T \\ 0 & U_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & P_1^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ P_1 \frac{a_{21}}{a_{11}} & L_1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12}^T \\ 0 & U_1 \end{pmatrix}$$

$$= \quad \bar{P}^T \quad L \quad U \Rightarrow \bar{P} P A = LU$$

$L = I, \quad P = I$

for $j = 1:n-1$

  if $A(j,j) = 0 \Rightarrow$ find $k > j$ s.t. $A(k,j) \neq 0$

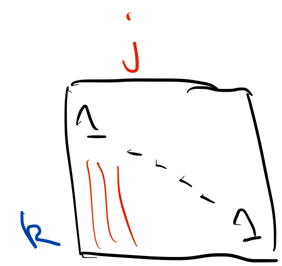
  | swap rows j and k of A, L; update P

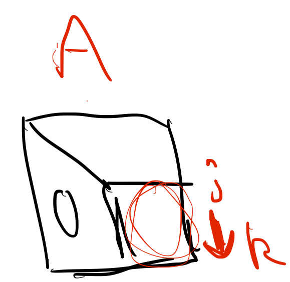
  $L(j+1:n, j) = A(j+1:n, j) / A(j,j)$

  $A(j+1:n, j) = 0$

  $A(j+1:n, j+1:n) \mathrel{-}= L(j+1:n, j)\, A(j, j+1:n)$

$U = A$





$$A = \begin{pmatrix} \delta & 1 \\ 1 & 1 \end{pmatrix} \qquad \delta \text{ small}$$

$$A = \begin{pmatrix} 1 & 0 \\ 1/\delta & 1 \end{pmatrix}\begin{pmatrix} \delta & 1 \\ 0 & 1-1/\delta \end{pmatrix} = LU$$

$$\|A\|_1 = 2 \qquad \|L\|_1 \approx 1/\delta \qquad \|U\|_1 \approx 1/\delta$$

$$Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = b \qquad \begin{pmatrix} 1 & 0 \\ 1/\delta & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 1 \\ -1/\delta \end{pmatrix}$$

$$\begin{pmatrix} \delta & 1 \\ 0 & \underbrace{1-1/\delta}_{\approx -1/\delta} \end{pmatrix} x = \begin{pmatrix} 1 \\ -1/\delta \end{pmatrix} \Rightarrow x \approx \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$Ax = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ \delta & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1-\delta \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1-\delta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow x \approx \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$Ax \approx \begin{pmatrix} 1-\delta \\ 0 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 0 \end{pmatrix} = b$$

Idea: pivot with largest entry

Gaussian Elimination with Partial Pivoting (GEPP)

for $j = 1:n-1$

pick $k = \underset{\ell \in \{j, \dots, n\}}{\arg\max} |A(\ell, j)|$ (*)

$p[j] = k$ P swaps rows j and k of I

swap $A(j,:)$ and $A(k,:)$

$A(j+1:n, j) = \underline{A(j+1:n, j) / A(j,j)}$ $L(:,j)$

$A(j+1:n, j+1:n) \mathrel{-}= A(j+1:n, j)\, A(j, j+1:n)$ S

L in lower tri, U upper tri

(*) $|L(i,j)| \le 1 \implies \|L\|_1 = O(n)$

Blocking

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$-A_{21}A_{11}^{-1}\cdot(\text{first row})$$
$$= -A_{21}A_{11}^{-1}(A_{11} \quad A_{12})$$
$$= -A_{21} \quad -A_{21}A_{11}^{-1}A_{12}$$

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22}-A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$$

S = schur complement

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$$

want this factorization

1. $A_{11} = L_{11}U_{11}$
2. $A_{21} = L_{21}U_{11}$ ✓ $\quad L_{21} = A_{21}U_{11}^{-1}$
3. $A_{12} = L_{11}U_{12}$ ✓ $\quad U_{12} = L_{11}^{-1}A_{12}$
4. $A_{22} = L_{21}U_{12} + L_{22}U_{22}$

$$\underbrace{A_{22} - L_{21}U_{12}}_{} = L_{22}U_{22}$$

$$S = A_{21}U_{11}^{-1}L_{11}^{-1}A_{12} = A_{21}A_{11}^{-1}A_{12}$$

Problem: pivoting is BLAS 1

Deferred pivoting

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Tournament pivoting

$$\begin{pmatrix} A_{11} & \cdots & A_{1B} \\ A_{21} & & \vdots \\ \vdots & & \vdots \\ A_{p1} & \cdots & A_{pB} \end{pmatrix}$$

Problem: $\|U\|$ with GEPP

GECP: search over rows and cols

$$PAQ = LU$$