Sep 18, 2020

Solving square linear systems of equations

$$n\overset{n}{\boxed{A}}\; n\overset{1}{\boxed{x}} = n\overset{1}{\boxed{b}}$$

$\cancel{A^{-1}b}$ $\Rightarrow$ $\cancel{\text{inv}(A)\cdot b}$

just for math notation

bad!

$x: Ax = b$

1) destroy structure
2) slower than GE
3) not as stable

We will study a factorization approach:

$$PA = LU$$

perm. matrix

that is <u>backward stable</u>

"Direct" solver

$PA = LU$, A nonsingular

$$Ax = b$$
$$PAx = Pb$$
$$LUx = Pb$$

① $c = Pb$ $O(n)$

② $Ly = c$ forward sub $O(n^2)$

③ $Ux = y$ backward sub $O(n^2)$

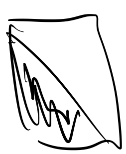
$$\begin{array}{ccc} L & y & c \\ \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} & \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = & \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \end{array}$$

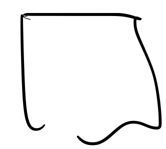
$1 \cdot y_1 = 1 \Rightarrow y_1 = 1$

$2 \cdot y_1 + y_2 = 1 \Rightarrow y_2 = -1$

$3y_1 + 2y_2 + y_3 = 3$

$3 - 2 + y_3 = 3$ $y_3 = 2$





```
y = 0
for i = 1:n
    s = c_i
    for j = 1:i-1
        s -= l_ij y_j
    y_i = s / l_ii
```

$2(0+1+2+\dots+n-1)$
$= O(n^2)$

```
x = 0
for i = n:-1:1
    s = y_i
    for j = i+1:n
        s -= u_ij x_j
    x_i = s/u_ii
```

Still need $PA = LU$ (ignore P for now): Strategy:

$$A = \begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix} \xrightarrow{L_1 A} \begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & x & x \end{pmatrix} \xrightarrow{L_2(L_1 A)} \begin{pmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{pmatrix}$$

$L_2 L_1 A = U \Rightarrow A = L_1^{-1} L_2^{-1} U$ $\quad L_1, L_2$ are unit lower tri

$$L L^{-1} = I \qquad \underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ & & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} & z_{\cdot j} & \end{pmatrix}}_{z} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

$0 = L_{11} z_1 = z_1 = 0$

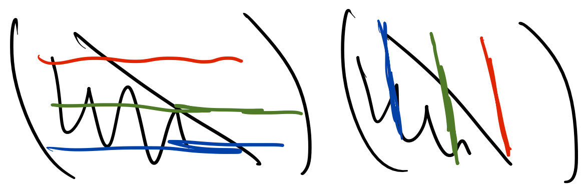
$0 = L_{21} z_1 + z_2 = 0$

$\vdots$

$1 = L_{j1} z_1 + \cdots + L_{jj} z_j$

$1 = z_j$

$(L_1^{-1} L_2^{-1})_{ij}$



$i < j \Rightarrow 0$

$i = j \Rightarrow 1$

$i > j \Rightarrow$ possibly nonzero ✓

① What are $L_1, \dots, L_{n-1}$? ✓

② Efficiently compute $L_1^{-1} \cdots L_{n-1}^{-1}$?

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}$$

$-2 \cdot$ (first row) + second row

$-3 \cdot$ (first row) + third row

$$L_1 A = \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -11 \end{pmatrix}$$

$-2 \cdot$ (second row) + third row

$$L_2 L_1 A = \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix}$$

Gauss transformation

$$L_1 A = \left(I - \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} (1\ 0\ 0)\right) A = (I - \tau_1 e_1^T) A$$

$$L_2(L_1 A) = \left(I - \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} (0\ 1\ 0)\right)(L_1 A) = (I - \tau_2 e_2^T)(I - \tau_1 e_1^T) A$$

$$I - \tau_j e_j^T = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} - \begin{pmatrix} & 0 & \\ & \vdots & \\ & 0 & \\ & \vdots & \end{pmatrix} = \text{unit lower triangular}$$

$$I - \tau_2 e_2^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$(I - \tau_j e_j^T)(I + \tau_j e_j) = I - \tau_j e_j^T \tau_j e_j \qquad (0 \cdots 0\ 1\ 0 \cdots 0) \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = 0$$

$$L_{n-1}\cdots L_1 A = U \Rightarrow A = L_1^{-1}\cdots L_{n-1}^{-1}U$$

$$(1\ 0 \cdots 0)\begin{pmatrix}0\\0\\x\\\vdots\\x\end{pmatrix} = 0$$

$$(I+\tau_1 e_1^T)(I+\tau_2 e_2^T) = I + \tau_1 e_1^T + \tau_2 e_2^T + \tau_1 e_1^T \tau_2 e_2$$

$$L_1^{-1}\cdots L_{n-1}^{-1} = I + \tau_1 e_1^T + \tau_2 e_2^T + \cdots + \tau_{n-1} e_{n-1}^T$$

$$\left[\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix} + \begin{pmatrix}0 & \\ x & \\ \vdots & 0\\ x & \end{pmatrix} + \begin{pmatrix} & 0 & \\ & 0 & \\ 0 & x & 0\\ & \vdots & \\ & x & \end{pmatrix} + \cdots\right] \quad \begin{pmatrix}1 & & 0\\ & \ddots & \\ \text{M} & & 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{pmatrix} = \left(I + \begin{pmatrix}0 & 0 & 0\\ 2 & 0 & 0\\ 3 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 2 & 0\end{pmatrix}\right)\begin{pmatrix}1 & 4 & 7\\ 0 & -3 & -6\\ 0 & 0 & 1\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 2 & 1\end{pmatrix}$$

$L = I$
$U = A$ } maintain updates

for $j = 1 : n-1$
  for $i = j+1 : n$
    $L(i,j) = U(i,j) / U(j,j)$ multiplier in Gauss transformation in ith row
    for $k = j+1 : n$
      $U(i,k) = U(i,k) - L(i,j) U(j,k)$
subtracting jth row from ith row multiplier $L(i,j)$

return L, triu(U)

Operation count: $O(n^3)$