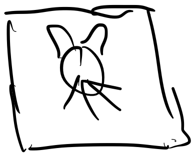
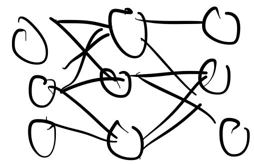
Sept. 11, 2020

Problem we want to solve with some input

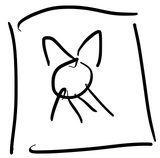
$(A, b) \longrightarrow x : Ax = b$

$\Big(\big\{ \big($  , cat $\big) \big\}$  $\Big) \longrightarrow$ opt. NN weights

Suppose we have perfect algorithm

Is this enough?

errors in input? A is "truth" A+E is measured

 , dog

Perturbations of input cause changes in output

We describe this with conditioning

A problem + input is

well-conditioned: small input changes $\Rightarrow$ small output changes

ill-conditioned: small large

What is "small" or "large" want x, get $\hat{x}$

Absolute error: $\|\hat{x} - x\|$

Relative error: $\|\hat{x} - x\| / \|x\|$ need to know $\|x\|$

sometimes can approx RE with $\|\hat{x} - x\| / \|\hat{x}\|$

In Num. lin alg., the choice of norm is important

$$\frac{\|\hat{x} - x\|_\infty}{\|x\|_2} \Rightarrow \frac{\|\hat{x} - x\|_2}{\|x\|_2}$$

$\max_i \|\hat{x}_i - x_i\| \leq \|x\|_2$

$(\|\hat{x} - x\|_\infty \leq \|\hat{x} - x\|_2)$

Example: matvec

$$(A, x) \longrightarrow y = Ax$$
$$(A, x+e) \longrightarrow \hat{y} = A(x+e)$$

Absolute error: $\|\hat{y} - y\| = \|A(x+e) - Ax\| = \|Ae\|$

Relative error: $\|Ae\| / \|y\| = \|Ae\| / \|Ax\|$

how bad? $e \to \infty \Rightarrow RE \to \infty$ $\quad e \to 0 \Rightarrow RE \to 0$

$\|x+e-x\|$

$$\frac{\|Ae\|}{\|Ax\|} \Big/ \frac{\|e\|}{\|x\|} = \frac{\|Ae\|}{\|e\|} \cdot \frac{\|x\|}{\|Ax\|}$$

Worst-case: $\left(\sup_e \frac{\|Ae\|}{\|e\|} \right) \frac{\|x\|}{\|Ax\|} = \|A\| \frac{\|x\|}{\|Ax\|}$

$\frac{\|A\| \cdot \|x\|}{\|Ax\|}$ is the condition number of the problem (input: A, x)

More general formulation

$f : X \to Y$ relative error: $\frac{|f(x+h) - f(x)|}{|f(x)|}$ ⊛

$x \to x+h$

f diff

⊛ $\approx \frac{|f(x) + f'(x)h - f(x)|}{|f(x)|} = \frac{|h||f'(x)|}{|f(x)|}$

$= \underbrace{\frac{|h|}{|x|}}_{\text{relative input}} \underbrace{\frac{|f'(x)||x|}{|f(x)|}}_{\text{condition number}}$

The condition number is

$$k(f, x) = \lim_{h \to 0} \sup_{\|e\| \leq h} \frac{\|f(x+e) - f(x)\|}{\|f(x)\|} \Big/ \frac{\|e\|}{\|x\|}$$

$$\sup_{e} \frac{\|Ae\|}{\|e\|} \frac{\|x\|}{\|Ax\|} = \frac{\overset{f'(x)}{\|A\|} \|x\|}{\|Ax\|}$$

$f_A : \mathbb{R}^m \to \mathbb{R}^n$

$f_A(x) = Ax$

$$\frac{\partial (f_A)_i}{\partial x_j} = \frac{\partial \sum A_{ij} x_j}{\partial x_j} = A_{ij}$$

Example: solving linear system $\quad A \in \mathbb{R}^{n\times n}$ nonsing.

$$(A, b) \xrightarrow{f} x : Ax = b \quad (x = A^{-1}b)$$

$$(A+\delta A, b) \xrightarrow{f} x + \delta x : (A+\delta A)(x+\delta x) = b$$

$$(A+\delta A)(x+\delta x) = b$$

$$= Ax \qquad = b$$

$$\Rightarrow \delta A x + A\delta x + \cancel{\delta A \delta x} = 0$$

$$\Rightarrow A\delta x = -\delta A x$$

$$\delta x = -A^{-1}(\delta A x)$$

$$\|\delta x\| = \|A^{-1}(\delta A x)\|$$

$$\le \|A^{-1}\| \|\delta A\| \|x\|$$

rel output change $\quad$ rel input change

$$\frac{\|\delta x\|}{\|x\|} \Big/ \frac{\|\delta A\|}{\|A\|}$$

$$\le \frac{\|A^{-1}\| \cancel{\|\delta A\|} \cancel{\|x\|} \|A\|}{\cancel{\|x\|} \cancel{\|\delta A\|}}$$

$$= \|A^{-1}\| \cdot \|A\|$$

$\kappa(A) =$ "the condition number of A"

Note 1: cond num depends on problem

Note 2: upper bound

$$\|\delta x\| = \|A^{-1}\delta A x\| \le \|A^{-1}\|\,\|\delta A\|\,\|x\|$$

$$x = A^{-1}b$$

can show $\exists\, \delta A$ s.t. equality

$$\kappa(A) = \|A^{-1}\|\,\|A\|$$

Note 3: What is $\kappa_2(A) = \|A^{-1}\|_2\|A\|_2 = \sigma_1(A)\sigma_1(A^{-1})$

$A = U\Sigma V^T$ $\quad A^{-1} = V\Sigma^{-1}U^T$ $\quad$ SVD of A^{-1}?

$$\Sigma^{-1} = \begin{pmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{pmatrix} \qquad A^{-1} = V\Pi\,\Pi^T\Sigma^{-1}\Pi\,\Pi^T U^T$$

$$\|A^{-1}\|_2 = 1/\sigma_n(A)$$

$$\kappa_2(A) = \frac{\sigma_1(A)}{\sigma_n(A)}$$