Sep 9, 2020

The Singular Value Decomposition (SVD)

$A \in \mathbb{R}^{m\times n}$ $m \geq n$. We can write $A = U\Sigma V^T$

right sing. vecs

$V^TV = I$

$U^TU = I$ $\Sigma = \mathrm{diag}(\overbrace{\sigma_1, \sigma_2, \dots, \sigma_n}^{\text{singular values}})$

$\sigma_1 \geq \sigma_2 \cdots \geq \sigma_n \geq 0$

$\|A\|_2 = \|\cancel{U}\Sigma \cancel{V}^T\|_2 = \|\Sigma\|_2 = \sup_{\|x\|_2=1} \|\Sigma x\|_2^2 \quad \sum_{i=1}^{n} x_i^2\sigma_i^2 = \sigma_1$

$\|A\|_F = \|\Sigma\|_F = \sqrt{\sum \sigma_i^2}$

SVD existence (sketch)

$\|A\|_2 = \sigma_1 = \sup_{\|x\|_2=1} \|Ax\|_2$ Let $\|Av_1\|_2 = \sigma_1$ $\|v_1\|_2 = 1$

$u_1 = Av_1/\sigma_1$ $\|u_1\|_2 = 1$

$\bar{U} = {}_m\big[\overset{1}{u_1}\ \overset{m-1}{U_2}\big]$ $\bar{V} = {}_n\big[\overset{1}{v_1}\ \overset{n-1}{V_2}\big]$

$$\bar{U}^T A \bar{V} = \begin{array}{c|cc} & 1 & n-1 \\ \hline 1 & \overset{\sigma_1}{\cancel{u_1^T A v_1}} & 0\ \cancel{w^T} \\ m-1 & \overset{0}{\cancel{U_2^T A v_1}} & B \end{array}$$

$u_1^T A v_1 = \sigma_1 u_1^T u_1 = \sigma_1$

$U_2^T A v_1 = \sigma_1 U_2^T u_1 = 0$

$$\Bigg\| \underbrace{\begin{pmatrix} \sigma_1 & w^T \\ 0 & B \end{pmatrix}}_{S} \begin{pmatrix} \sigma_1 \\ w \end{pmatrix} \Bigg\|_2^2 = \left\| \begin{matrix} \sigma_1^2 + w^T w \\ Bw \end{matrix} \right\|_2^2 \geq \sigma_1^2 + w^T w$$

$\|S\|_2 = \|\bar{U}^T A \bar{V}\|_2 = \|A\|_2 = \sigma_1 \Rightarrow$ ⊛ $w^T w = \|w\|_2^2 \leq 0$

$\Rightarrow w = 0$

Assume $B = \underline{U}\,\underline{\Sigma}\,\underline{V}^T$ (induction)

$$A = \bar{U} \begin{pmatrix} \sigma_1 & 0 \\ 0 & B \end{pmatrix} \bar{V}^T = \underbrace{\bar{U} \begin{bmatrix} 1 & 0 \\ 0 & \underline{U} \end{bmatrix}}_{U} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \underline{\Sigma} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & \underline{V}^T \end{bmatrix} \bar{V}^T}_{V^T}$$

unique?

⊛ $\|S\|_2 = \sup_{\|x\|_2=1} \|Sx\|_2$ $\leq \sigma_1^2$

$\geq \sigma_1^2$

Geometry of SVD

$$A = U\Sigma V^T$$

$$y = Ax$$

① $y_1 = V^T x$ rotation/reflection

② $y_2 = \underline{\Sigma} y_1$ stretching

③ $y = U y_2$ rotation/reflection

Relationship with eigendecomposition

$V^T = V^{-1}$

$A = U \Sigma V^T$

$A^T A = V \Sigma \overset{I}{\cancel{U^T}} U \Sigma V^T = V \Sigma^2 V^T$ eigendecomposition

$A A^T = U \Sigma^2 U^T$

Last time: $\|A\|_2 = \sqrt{\lambda_{max}(A^T A)} = \sigma_1$

B symmetric $\Rightarrow B = X \Lambda X^T$ $X^T X = I$

$$= \underbrace{X}_{U} \underbrace{|\Lambda|}_{\Sigma} \underbrace{\begin{pmatrix} \operatorname{sign}(\lambda_1) & & \\ & \ddots & \\ & & \operatorname{sign}(\lambda_n) \end{pmatrix} X^T}_{V^T}$$

Rank/range/null space

rank(A) = # (nonzero singular values)

$\mathrm{rank}(A) = \mathrm{rank}(U \Sigma V^T) = \mathrm{rank}(\Sigma) = \dim(\{\Sigma x \mid x \in \mathbb{R}^n\})$

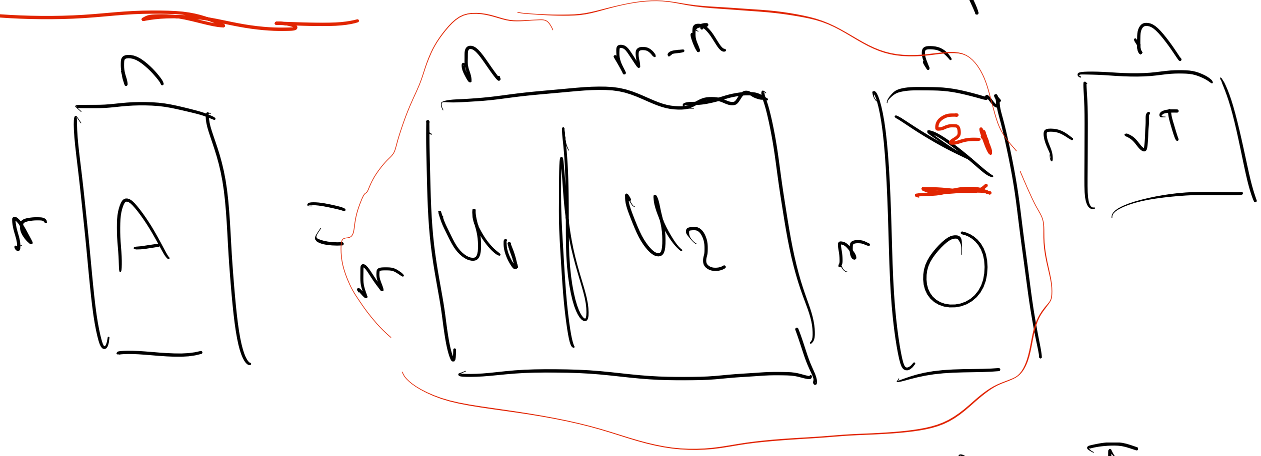
= # nonzeros on diag

$\mathrm{rank}(A) = r$

$\mathrm{range}(A) = \mathrm{span}\{u_1, \ldots, u_r\}$

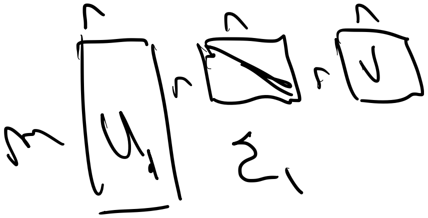
$\mathrm{null}(A) = \mathrm{span}\{v_{r+1}, \ldots, v_n\}$

Thin SVD (reduced / compact)



$$= (U_1\ U_2)\begin{pmatrix}\Sigma_1\\ 0\end{pmatrix}V^T$$

$$= U_1\Sigma_1V^T$$



$U_1^TU_1 = I_{n\times n}$

$$= \sum_{i=1}^{n}\sigma_i u_i v_i^T$$

rank r $\Longrightarrow$ $A = \sum_{i=1}^{r}\sigma_i u_i v_i^T$

SVD as low-rank approx (Eckart-Young)

$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ $k < \text{rank}(A)$ $A = U\Sigma V^T$

$$\min_{\text{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}$$

also true for $\|A - B\|_F = \|A - A_k\|_2 = \sqrt{\sum_{i=k+1}^{n} \sigma_i^2}$

$\|x - y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 - 2x^T y$ $X = \begin{bmatrix} x_1^T \\ \vdots \\ x_m^T \end{bmatrix}$

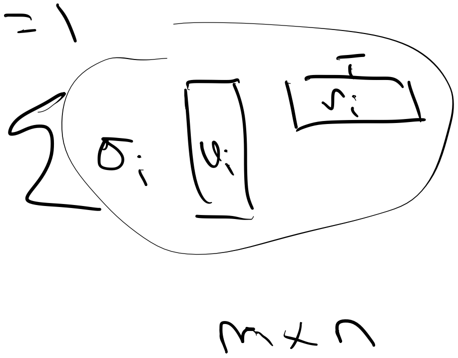
Dataset $x_1, \dots, x_m \in \mathbb{R}^n$ $\|x_i\|_2^2 = 1$

similarity $\propto -\|x - y\|_2^2 = -(1 + 1 - 2x^T y) = c + c_1 x^T y$

$\min_{\text{rank}(B)=k} \|XX^T - B\|_{2,F}$ $B = \sum_{i=1}^{k} \sigma_i^2 u_i u_i^T$

$X = U\Sigma V^T$

$A = \sum_{i=1}^{n} \sigma_i u_i v_i^T$ $u_i \in \mathbb{R}^m$ $v_i \in \mathbb{R}^n$



$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$

rank-k approx: $O(km + kn)$ k small $<< O(mn)$