Sept. 7, 2020

Norms are how we measure length $\|\cdot\|$

1. Length of a vector: $\|x\|$
2. Length of the difference: $\|x-y\| \Rightarrow$ distance

Ex: relative error $\dfrac{\|\hat{x}-x\|}{\|x\|}$ dist

relative to size

$\|\cdot\|$ is a norm if

(i) $\|x\| \geq 0$ $\quad \|x\|=0 \iff x=0$

(ii) $\|\alpha x\| = |\alpha|\,\|x\|$

(iii) $\|x+y\| \leq \|x\| + \|y\|$ triangle ineq.

x, y elements of some vector space

start $\mathbb{R}^n$

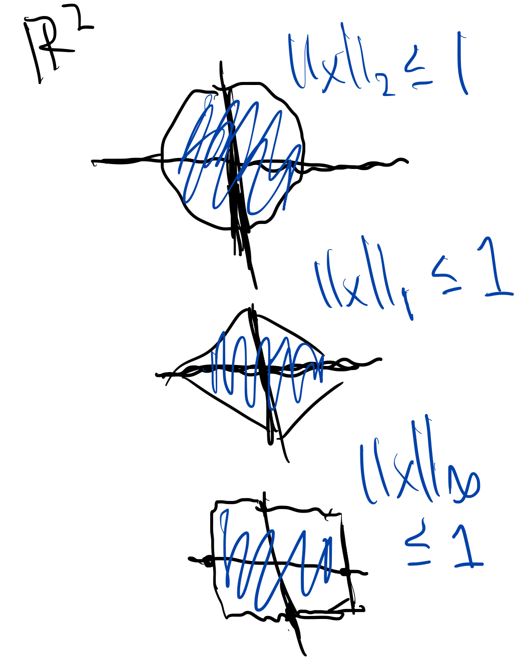
Most common norms on $\mathbb{R}^n$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$ "2-norm" "Euclidean norm"

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$ "1-norm"

$$\|x\|_\infty = \max_i |x_i|$$ "infinity norm" "max norm"

"p-norm" $\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ $\quad \|x\|_p$



Isometries

- permutations $\quad \|\Pi x\|_p = \|x\|_p$
- orthogonal matrices for $\|\cdot\|_2$

$$\|x\|_2^2 = \sum x_i^2 = x^T x$$

$$\|Qx\|_2^2 = (Qx)^T(Qx) = x^T Q^T Q x = x^T x = \|x\|_2^2$$

Norm equivalence $\|\cdot\|$, $\|\cdot\|'$ on finite-dim VS V

$\exists$ constants c_1, c_2 s.t.

$$c_1 \|x\| \leq \|x\|' \leq c_2 \|x\| \quad \forall x \in V$$

Example: $\|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty \quad x \in \mathbb{R}^n$ $\quad c_1 = 1$, $c_2 = \underline{n}$

Proof:
$$\|x\|_\infty = \max_i |x_i|$$
$$\leq \sum_{i=1}^n |x_i| = \|x\|_1$$
$$\leq n \max_i |x_i| = n \|x\|_\infty$$

"standard inner product" on $\mathbb{R}^n$ $\langle x, y\rangle = x^T y = \sum_{i=1}^{n} x_i y_i$
"dot product"

IP: $\langle x, x\rangle \geq 0$ $\quad \langle x, x\rangle = 0 \iff x = 0$

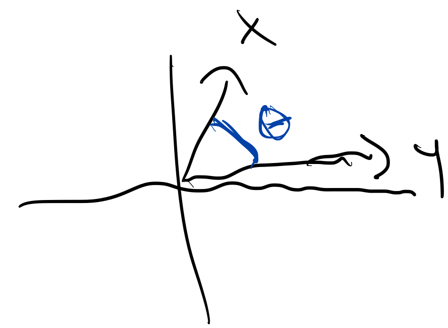
$\langle \alpha x, y\rangle = \alpha \langle x, y\rangle$ $\qquad \langle x, y\rangle = \overline{\langle y, x\rangle}$

$= \langle y, x\rangle$ over $\mathbb{R}$

$\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

IP $\Rightarrow$ norm by $\|x\| = \sqrt{|\langle x, x\rangle|}$

$\Rightarrow$ angles $\cos\theta = \dfrac{\langle x, y\rangle}{\|x\|\,\|y\|}$



Cauchy-Schwarz inequality

$|\langle x, y\rangle|^2 \leq |\langle x, x\rangle|\,|\langle y, y\rangle| = \|x\|^2 \|y\|^2$

$|\cos\theta| = \dfrac{|\langle x, y\rangle|}{\|x\|\,\|y\|} \leq \dfrac{\|x\|\,\|y\|}{\|x\|\,\|y\|} = 1$

$|x^T y|^2 \leq \|x\|_2^2 \|y\|_2^2 \implies |x^T y| \leq \|x\|_2 \cdot \|y\|_2$

Hölder ineq: $|x^T y| \leq \|x\|_p \|y\|_q$ $\quad \frac{1}{p} + \frac{1}{q} = 1$ $\quad (p = q = 2)$

What about matrices? $A \in \mathbb{R}^{m \times n}$

treat as mn-dim VS

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}$$ "Frobenius norm"

induced matrix norms (operator norms)

$$\|A\| = \sup_{\|x\| \neq 0} \frac{\|Ax\|'}{\|x\|}$$

$\|\cdot\|'$ - norm on $\mathbb{R}^m$

$\|\cdot\|$ norm on $\mathbb{R}^n$

$$\|\cdot\|' = \|\cdot\| = \|\cdot\|_p$$

y $\|y\|_p = 1$

$$\|A\|_p = \sup_{\|x\|_p \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\|_p \neq 0} \left\| A \left(\frac{x}{\|x\|_p} \right) \right\|_p$$

$$= \sup_{\|x\|_p = 1} \|Ax\|_p$$

$$\|A\|_1 = \sup_{\|x\|_1 = 1} \|Ax\|_1 = \|x_1 a_1 + \dots + x_n a_n\|_1$$

$$\leq \sum_i |x_i| \, \|a_i\|_1 \qquad \|x\|_1 \leq 1$$

$$\leq \max_i \|a_i\|_1$$

$$x = e_j \quad j = \operatorname{arg\,max}_j \|a_j\|_1 \Rightarrow \|A\|_1 = \max_i \|A(:,i)\|_1$$

"max abs col sum"

$$\|A\|_\infty = \max_j \|A(j,:)^T\|_1 \quad \text{max abs. row sum}$$

induced norms are "consistent" / "sub-ordinate"

$$\|Ax\|_p \leq \|A\|_p \|x\|_p \qquad x = 0 \checkmark$$

$$\|Ax\|_p / \|x\|_p \leq \sup_{\|x\|_p \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \|A\|_p$$

$$\|Ax\|_2 \leq \|A\|_F \|x\|_2$$

induced norms are submultiplicative

$\|AB\|_p \le \|A\|_p \|B\|_p \qquad A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{n \times k}$

$\|AB\|_F \le \|A\|_F \|B\|_F$

max-norm $\quad \max_{i,j} |a_{ij}|$ is not!

$$C = \overset{A}{\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}} \overset{B}{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}} \qquad \|A\| = 2 \quad \|B\| = 1$$

$c_{11} = 4 \quad \|C\| = 4$

Isometries

$\|A\|_1$ = max abs col sum $\qquad \|\Pi_1 A \Pi_2\|_1 = \|A\|_1$

$\|Q_1 A Q_2\|_2 = \|A\|_2 \quad Q_i^T Q_i = I, \quad i = 1, 2$

$\|Q_1 A Q_2\|_F = \|A\|_F$

matrix 2-norm "spectral norm"

$$\|A\|_2^2 = \sup_{\|x\|_2=1} \|Ax\|_2^2$$

$x^T A^T A x$

$A^T A$ symm $\Rightarrow$ $A^T A = V \Lambda V^T$

$V^T V = I \quad \Lambda = (\diagdown)$

$$x^T A^T A x = x^T V \Lambda V^T x$$

$$\{\|x\|_2 = 1\} = \{\|Vx\|_2 = 1\} \qquad y = V^T x$$

$$\sup_{\|y\|_2=1} y^T \Lambda y = \sup \sum_{i=1}^{n} y_i^2 \lambda_i \le \lambda_{max}(A^T A)$$

choose $y = e_1 \Rightarrow \lambda_1$

$$V V^T x = V e_1 \Rightarrow x = V e_1 = v_1$$

$$\|A\|_2 = \sqrt{\lambda_{max}(A^T A)}$$