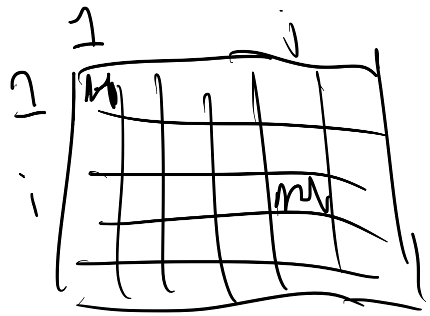
Sep 2, 2020

What is a matrix? 2-d array of numbers

A 

A_{ij}

a vector (1-d)

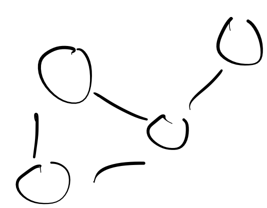
x 

x_i

<u>Linear algebra</u>: $T: V \to W$

A matrix represent linear map w/r/t choice of basis

Matrix from data: A_{ij} = # times that words i and j appear in same sentence

Matrix from graph: 

$$A_{ij} = \begin{cases} 1 & (i,j) \text{ exists} \\ 0 & \text{o/w} \end{cases}$$

Matrix transformation in NN: $y_i \approx c^T \sigma(W_2 \sigma(W_1 x_i + b_1) + b_2)$

PDE discretized

We'll start from matrices and some LA knowledge

$$A(x+y) = Ax + Ay$$

Notation

$\begin{array}{cc} & n \\ m & \boxed{A} \end{array}$ matrix $\qquad \begin{array}{cc} & 1 \\ n & \boxed{x} \end{array}$ vector $\qquad \begin{array}{cc} & 1 \\ 1 & \boxed{\alpha} \end{array}$ scalar

entry A_{ij} $\qquad x_i$

$A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$ finite-dim

$A \in \mathbb{C}^{m \times n}$, $x \in \mathbb{C}^n$

① linear systems

$$Ax = b$$

$$\begin{array}{cc} & n \\ n & \boxed{A} \end{array} \begin{array}{cc} & 1 \\ n & \boxed{x} \end{array} = \begin{array}{cc} & 1 \\ n & \boxed{b} \end{array}$$

② least squares

$$\min_x \|Ax - b\|_2^2$$

$$\begin{array}{cc} & n \\ m & \boxed{A} \end{array} \begin{array}{cc} & 1 \\ n & \boxed{x} \end{array} \approx \begin{array}{cc} & 1 \\ m & \boxed{b} \end{array}$$

③ eigenvalue

$$Ax = \lambda x$$

$$\begin{array}{cc} & n \\ n & \boxed{A} \end{array} \begin{array}{cc} & 1 \\ n & \boxed{x} \end{array} = \lambda \begin{array}{c} 1 \\ \boxed{x} \end{array}$$

Algorithms/computations for solving these

(A) accuracy
- guarantee correct solution (exact arith)
- stable w/r/t perturbations
- approximate answer OK

(B) structure
- dense vs. sparse
- symmetric or not, pos. def.
- kernel $K_{ij} = f(\|x_i - x_j\|_2)$

(c) efficiency — $O(n^2)$ vs. $O(n^3)$

$O(\#\text{ non zero entries})$

accounting for computer arch.

One recurring theme: factorizations/decompositions

Example: $A = PLU$ $\qquad$ $A = QR$

perm. mtx

I with rows permuted

$Q^TQ = I$

$A = XY$, $\qquad$ $A = XYZ$

Goal: literate + comfortable with matrix comps to

(i) use tools more effectively

(ii) help develop new tools

Admin

CMS
- 7 HWs (7 x 10%)
- 1 final exam: take-home (25%)
- participation 5% attend lecture

Goal: HW, exam, lecture, reading (4x)

web site

Matrix multiplication

$$\begin{array}{cc} & n \\ m & \boxed{\;A\;} \end{array} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \qquad \begin{array}{cc} & 1 \\ n & \boxed{x} \end{array} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$y = Ax$ mat-vec $\qquad$ y m-dim

Def: $y_i = \sum_{j=1}^{n} A_{ij} x_j \quad i = 1, \ldots, m$

Dense: $O(mn)$ $\qquad$ square: $O(n^2)$

Sparse: $O(\mathrm{nnz}(A))$

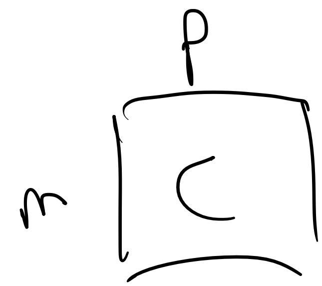
$A(x+y) = Ax + Ay \qquad A(\alpha x) = \alpha A x$

$$y = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

= linear comb. of cols of A

matrix-matrix mult (mat-mul)

$C = AB$



<u>Def</u>: $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \quad i = 1, \dots, m, \; j = 1, \dots, p$

$O(mnp) \quad m = n = p \quad O(n^3)$

$A(B+G) = AB + AG \qquad A(\alpha B) = \alpha AB$

associative: $(AB)G = A(BG)$

not generally commutative: $AB \neq BA$

$$C = \begin{bmatrix} - & a_1^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | & & | \\ b_1 & \cdots & b_p \\ | & & | \end{bmatrix}$$

$$C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \qquad \underbrace{e_i^T C e_j}_{c_{ij}} = \underbrace{e_i^T A}_{a_i^T} \underbrace{B e_j}_{b_j}$$

$$e_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} j \qquad e_i^T = \begin{array}{c} \\ (0 \cdots 0 \end{array} \begin{array}{c} i \\ 1 \end{array} \begin{array}{c} \\ 0 \cdots 0) \end{array}$$

Mat-mul as sequence of mat-vec

$$\underbrace{C e_j}_{\substack{j^{th} \text{ col of } C \\ c_j}} = A(\underbrace{B e_j}_{b_j}) \qquad c_j = A b_j$$

linear comb of cols of A

$$C = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \sum_{i=1}^{n} a_i b_i^T$$

a_i b_i^T

$C = 0$

for $i = 1:m$

  for $j = 1:p$

    for $k = 1:n$

      $C_{ij} \mathrel{+}= A_{ik} B_{kj}$

column-by-column

$C = 0$

for $i = 1:m$

  for $k = 1:n$

    for $j = 1:p$

      $C_{ij} \mathrel{+}= A_{ik} B_{kj}$

$C(i,:) = A(i,:)B$