Midterm

(due: 2019-10-21)

You may (and should) use any references you wish: books, notes, MATLAB help, literature. But please do not consult with anyone outside the course staff. Templates for the required codes are provided in the class Github repository.

1: Identity plus Consider the matrix $A = I + ZZ^T$ where $Z \in \mathbb{R}^{n \times k}$, $k \ll n$.

2 pts Argue that A is symmetric and positive definite.

2 pts Give an $O(nk)$ time algorithm for computing $y = Ax$.

2 pts Given an economy SVD $Z = U\Sigma V^T$, show how to compute $\kappa_2(A)$ cheaply. How cheaply can it be done?

2 pts Given an economy QR decomposition $Z = QR$, show how to solve $Ax = b$ cheaply. How cheaply can it be done?

Please provide *both* a written description of your approach and code that satisfies the provided interfaces.

2: Daring derivatives Give codes to compute directional derivatives for each of the following

2 pts Differentiate $\|x\|_M^2$ with respect to changes in x and changes in M (for M spd).

2 pts Differentiate the solution to $(I + ZZ^T)x = b$ with respect to changes in Z and b, given a QR factorization of Z. Your code should require $O(nk + k^3)$ time.

2 pts Differentiate the Cholesky factorization $R^T R = M$ (*Hint*: differentiate the basic relation, then pre- and post-multiply by R^{-T} and R^{-1}, respectively).

Please provide *both* a written description of your approach and code that satisfies the provided interfaces.

3: Tridiagonal trouble Let T be a positive definite symmetric tridiagonal matrix

$$T = \begin{bmatrix} \alpha_1 & \beta_1 & & & & \\ \beta_1 & \alpha_2 & \beta_2 & & & \\ & \beta_2 & \alpha_2 & \beta_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ & & & & \beta_{n-1} & \alpha_n \end{bmatrix}$$

Assuming the tridiagonal is *unreduced* (i.e. none of the off-diagonal elements are zero), answer the following questions

2 pts Consider any block 2-by-2 decomposition of $S = T^{-1}$ with square diagonal blocks, and show that S_{12} and S_{21} are rank one.

2 pts Write a code to compute $\|T^{-1}\|_1$ efficiently and exactly (i.e. you should not use a technique like Hager's condition estimator). Please provide *both* a written description of your approach and code that satisfies the provided interfaces.

4: Conditioning [2 pts] For any operator norm, show that if $\|A^{-1}E\| < 1$ then

$$\kappa(A + E) \leq \kappa(A) \left(\frac{1 + \|A^{-1}E\|}{1 - \|A^{-1}E\|} \right) \leq \frac{\kappa(A)}{(1 - \|A^{-1}E\|)^2}$$

where κ denotes the condition number with respect to linear solves.