HW for 2019-11-04

(due: 2019-11-18)

You may (and should) talk about problems with each other and with me, providing attribution for any good ideas you might get. Your final write-up should be your own.

Consider $A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n$ with A symmetric and b not an eigenvector, and define

$$\phi(x) = \frac{1}{2} x^T A x - x^T b.$$

We wish to minimize ϕ subject to the constraint $x^T x = 1$, via the Lagrangian

$$L(x, \mu) = \frac{1}{2} x^T A x - x^T b - \frac{\mu}{2}(x^T x - 1).$$

1. Express x at a stationary point in terms of A, b, and μ.

2. Argue that the condition $\|x\|^2 = 1$, given the expression from the previous step, implies singularity of the matrix

$$\begin{bmatrix} (A - \mu I)^2 & b \\ b^T & 1 \end{bmatrix}$$

3. Eliminating the z variable in

$$\begin{bmatrix} (A - \mu I)^2 & b \\ b^T & 1 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = 0,$$

   show that μ satsifies the quadratic eigenvalue problem

$$\left[(A^2 - bb^T) - 2\mu A + \mu^2 I \right] y = 0.$$

   Solving via polyeig gives us all possible μ in $O(n^3)$ time.

4. At the constrained minimizer, we must satisfy that $v^T(A - \mu I)v$ is positive for all v s.t. $v^T x = 0$. Argue that this implies $\mu < \lambda_2(A)$, where $\lambda_2(A)$ is the second smallest eigenvalue of A.

5. Following the divide-and-conquer idea and the argument from the previous step, argue that ϕ has no more than three constrained minimizers.

6. Write a code to solve the problem, with the interface hw6solve(A, b) (returning the vector x). Please also write a test case to sanity check your solver!